2. The Basic equations and some properties of partial differential equations.

We will mostly concentrate on solving the two-dimentional incompressible flow problem in rectangular coordinates.

2.1 The Basic Equations

The fundamental equations for the 2-dimensional incompressible flow are the Navier-stokes equations and the continuity equations. In the absence of rotation, they are:

1) $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} =^{-} 1/_{\rho}\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$

2) $\frac{\partial v}{\partial v} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} =^{-} 1/_{\rho}\frac{\partial p}{\partial y} + \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2})$

3) $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

u, v velocities

p pressure

ρ density

ν viscosity

We can obtain numerical solutions for the set of equations, but for simplicity, as a first step, we will use the vorticity- stream function approach.

If we define the verticle component of the vorticity as $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, by cross-differentation (1) and (2), we obtain the **** equation

(4) $\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = \nu(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2})$

or

(5) $\frac{\partial \zeta}{\partial t} + \vec{v}(\nabla\zeta) = D\nabla^2\zeta = \frac{D\zeta}{Dt}$

The **** equation then consists of

an unsteady term $\frac{\partial \zeta}{\partial t}$

an advective** term $\vec{v} \cdot (\vec{v}\zeta)$

ves**** term $\nabla\nabla^2\zeta$

Since $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, we can define a streamfunction ψ such that $\frac{\partial \psi}{\partial x} = v$ and $\frac{\partial \psi}{\partial y} = -a$ The ver**** can then be expressed as

(6) $\nabla^2\psi = \zeta$ Poisson Equation

The verticiby** equation is classified as *parabolic*, which means that it is a critical value problem, where the solution is stepped out of some **** condition. On the other hand, the streamfunction equation (6) is *elliptic* or boundary-value problem which is usually solved by intensive methods.

The vorticity* equation can also be rewritten in what is called "conservative" form. By using the **continuity equation (5) can be rewritten as

(7) $\frac{\partial \zeta}{\partial t} + \nabla \cdot)\vec{(v)}\zeta) = \nu \nabla^2 \zeta$

$\vec{(v)}\nabla\zeta + \underline{\zeta(\nabla\vec{v})} \longleftarrow= 0$

The advantage of such a formulation will be discussed later.

Let's now perform a dimensional analysis of the vorticity** equation which will then give us an idea of each terms importance. $(u, v) \longrightarrow u$

$(x, y) \longrightarrow L$

$\zeta \longrightarrow^U /_L$

$t \longrightarrow^L /_U$ advective** line scale

Then (7) can be rewritten as

(8) $\frac{\partial \zeta\prime}{\partial t\prime} = -\nabla \cdot (\vec{\nabla}\prime\zeta\prime) + \frac{1}{Re}\nabla^2\zeta$

with $Re = \frac{UL}{\nu}$, Reynolds number.

High Reynolds number, Re $\gg$ d $\longrightarrow$ the *advective term is dominant and $\frac{L}{U}$ is the value which effectively characterizes the flow. But for low Reynolds number Re $\ll 1$, a charcetive*** line* depend at the difference*** is better

$t \longrightarrow \frac{\nu}{L^2}$

which gives... (9) $\frac{\partial \zeta\prime}{\partial t\prime} = -Re\nabla \cdot (\vec{v}\prime\zeta\prime) + \nabla^2\zeta$

As Re $\longrightarrow 0$, the *advective term drops out. The use of the appropriate time constant will minimize round off errors which is of importance.

We still have a complex set of equations out of a lot can be learned from one-dimensional equations.

The one dimensional advedrican* - difference** equation is

(10) $\frac{\partial \zeta}{\partial t} + \frac{\partial (u\zeta)}{\partial x} = \alpha \frac{\partial \zeta^2}{\partial x^2}$

ζ is here the vorticity**, but can also be any other advected or diffused flow property. u is generally a constant.

Another treep***** equation is simply

(11) $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \alpha\frac{\partial^2 u}{\partial x^2}$ $\underline{Burgers Equation}$

with the equivalent conservation form

(12) $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\frac{u^2}{2}) = \alpha\frac{\partial^2 u}{\partial x^2}$

2.2 Some Properties of Partial Differential Equations

Basics of $\underline{PDEs}$

Partial differentials are used to model a wide variety of physical phenomena. A number of properties can be used to distinguish the different type of differential equations.

$\underline{Example}$: $au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + f = 0$

$\underline{Order}$: The order of a PDE is the order of the highest occurring derivative. The order of the above example is 2. It is second order in x and y. Most equations derived from physical principles are usually 1st order in time and first or second order in space.

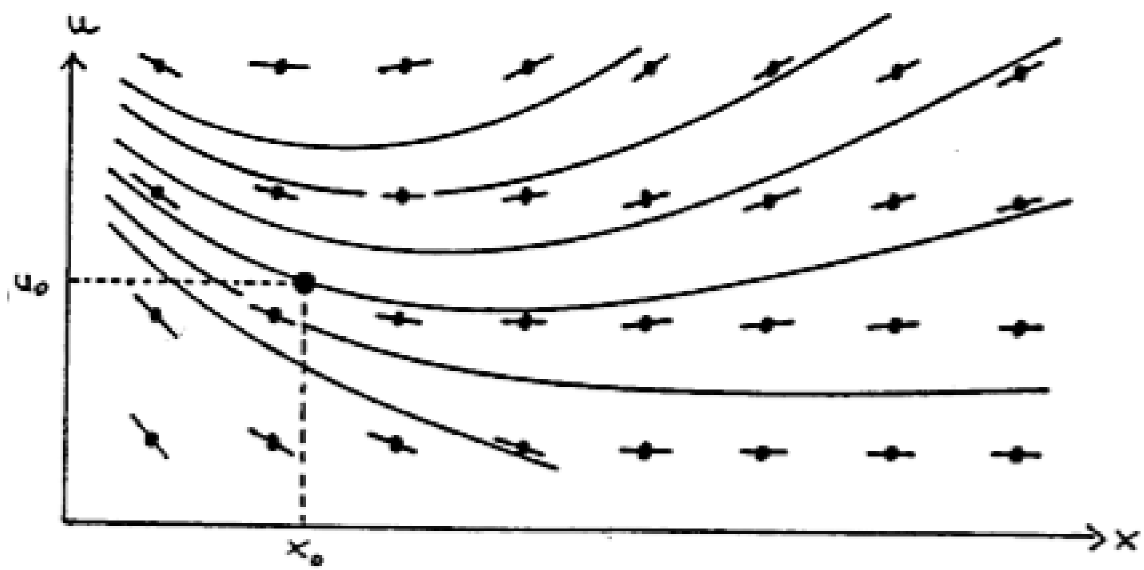
$\underline{Linear}$: The PDE is linear if none of its coefficients depend on the unknown function, i.e. a,b,c independent of u in the above example. Linear combinations of linear PDEs form another PDE. $w = \alpha a + \beta v$ is a solution of a PDE where both u and v are solutions.

The Laplace equation $a_{xx} + v_{yy} = 0$ is linear.

The Burger Equation $a_t + uu_x = 0$ is nonlinear.

(13) General Form: $T(u\prime, u, x) = 0$ with $u(x)$

The problem of finding the solution to u(x) from (13) is equivalent to draw a stream line in the (x,u) plane through a field of velocity vectors whose directions u' are given by (13) $u, x \longrightarrow u\prime$



Each curve shown is obviously a solution of the differential equation. Is our solution unique? Not necessarily.

Example : $u = xu\prime + (u\prime)^2$

$u = cx + c^2$ is a solution, but = - $\frac{x^2}{4}$ also a solution, but which cannot be obtained from a choice in the e****** c.

We then need to know when a unique solution does exist.

Theorem

Given the first order differential equation $u\prime = F(x.u)$

If F satisfies:

1) F is real, finite, single-valued, and continuous for all x, u.

2) $\frac{\partial F(x,u)}{\partial u}$ is real, finite, single-value and continuous.

Then there is a unique $u = g(x)$ which passes through any given point of R. (True of *linear* diff. equ.)

Then to make the solution unique, we find how* to prescribe* a "boundary condition" or specify a point which solution curve is supposed to pass through.

Linear DE

$u^n + b_{n-1}(x)u^{n-1} + b_{u-2}(x)u^{x-2}... + b_0(x)u$
$= r(x)$

b) Ordinary DE of a 1st order in two independence

(14) General Form $F(u_x, U_y, u, x, y)$ This equation implies that at each point in (x, y, u) space, the partial derivatives u_x, u_y are related, but they are NOT indirectly*** fixed. The orientation of the surface solution is NOT prescribed as a function of x,y,u. This additional degree of freedom requires boundary conditions at more than one point to ensure a unique solution. (assuming, of course, a LINEAR DE). u has to be specified at a curve C.

$\underline{S***}$

$au_x + bu_y + cv_x + dv_y = 0$

$Au_x + Bu_y + Cv_x + Dv_y = 0$

The solution is uniquely determined y, u, and v are specified on a curve C.

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