4. Stability properties of various time differencing schemes.

→ 4.1 - Applied to the advection* equation. $(\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x})$

The FTCS is a one-step, explicit, two-time level method.

One-Step: One calculator step is required to advance to a new time level.

Explicit: all the values on the right-side are known.

Two-time levels: Only two lines are involved in the calculation.

4. The Leap Frog scheme

The leap frog is centered in time which is unstable for the diffusive equation and adv doff equation but when applied to the adv. equation alone is stable.

$U_f^{n+1} = U_f^{n-1} - \lambda(U_{f+1}^n - U_{f-1}^n)$

$U_f = \Sigma A_k^n e^{ikxj}$

$$\boxed{A_k^{n+1} = A_k^{n-1} - A_k^n(2i\lambda\sin(k\Delta x))}$$

which can be rewritten in matrix form

using the trivial $A_k^n = A_k^n$. G is now the amplification factor. In this particular case of several time level, we are more cooditious* to fall back on a simple two-line level. (See homework 2).

Another way of presenting it is:

$A_k^{n+1} = C_k^n - A_k^n(2iA\sin(k\Delta x))$
$C_k^{n+1} = A_k^n$

The stability criteria for Von Neuman is $\mid \mu_i \mid \leq 1 + o(\Delta t)$ for all i where μ_i or the eigourale* of the matrix G. Another sufficient condition slightly schtline* is $\mid G \mid) \leq 1 + 0(\Delta t)$. Again, see Richtyer for a complete derivation.

Solving for the eiqueales*

$$\boxed{\mu_{\lambda 2} = \frac{\lambda}{2}(a \pm \sqrt{a^2 \pm 4})}$$

or

$$\boxed{\mu_{\lambda 2} = -i\lambda\sin(k\Delta x) \pm \sqrt{1 - \lambda^2\sin^2(k\Delta x)}}$$

-If $\lambda^2\sin^2(k\Delta x) > 1$, the square root term is then imaginary and,
$\mu_{\lambda 2} \;= +i(-\lambda\sin(k\Delta x)) \pm \sqrt{\lambda^2\sin^2(k\Delta x)-1}$
$\mid \mu_{\lambda 2}\mid^2 = 2\lambda^2\sin^2(\sin^2(k\Delta x) = 1 \pm 2\lambda^2\sin^2(k\Delta x)[1-\frac{1}{\lambda^2\sin^2(k\Delta x)}]^{\frac{1}{2}}$
$* \pm \rightarrow$ obviously >1, $\max(\mu_{\lambda 2}) > 1$

If $\lambda^2\sin^2 k\Delta x < 1$ (True for $\lambda < 1$), then the module of $\mid \mu_i \mid$ is given by
$\mid \mu_i \mid^2 = \lambda^2\sin^2(k\Delta x) + (1-\lambda^2 sin^2(k\Delta x)] = 1$
This obviously satisfies the requirement for stability provided again that $\lambda < 1$ $(\frac{c\Delta t}{\Delta x} < 1)$
Any numerical method for the *wascid equation which excludes an $\mid G\mid$ (or $S_R)(G), \parallel G \parallel) < 1$ exhibits an artificial damping. For any convergent method, the numerical damping error must of course, vanish as $\Delta x, \Delta t \rightarrow 0$. In this particular case of the leap-frog, the damping is equal to zero for $\mu C = ct^{***}$ and $\lambda < 1$.

The leap-frog for $\lambda = 1$ perpetuates the exact solution for all time given exact first time level solution. The vascid** equation $\frac{\partial\mu}{\partial t} = -c\frac{\partial\mu}{\partial x}$ is equivalent to say that
$\mu(x, t+\zeta) = \mu(x - c\zeta, t)$
if $\zeta = \Delta t$, then for $c = 1$
$(4) U_i^{n+1} = U_{i-1}^n$
over $2\Delta t$, the exact solution is
(5) $\mu_i^{n+1} = \mu_{i-2}^{n-1}$
Applications of the leap frog method given
(6) $U_c^{n+1} = U_i^{n-1} - U_{i+1}^n + U_{i-1}^n$
Given the correct starting values from (4)
$U_{i+1}^n = U_i^{n-1} and U_{i-1}^n = U_{i-2}^{n-1}$
then (6) is exactly equal to (5).

Two sets of values are required to start. If with an error, then the error will persist in the calculation.
The corresponding eigevales** to $\mu_{\lambda 2}$ are
$\vec{x_1}$ is not perpendicular to $\vec{x_2}$ but they are independent, and any vector can be expressed as a factor of $\frac{\vec{x_1}}{\vec{x_2}}$
$= (\alpha\vec{x_1} + \beta\vec{x_2})$

The $G_1, \vec{x_1}$ are associated with the steady part of the solution (Physical mode).

$G_2, \vec{x_2}$ part of the solution that changes sign every Form*** (Computational mode)

If we restrict a maniel** to the case where G' is equal for *** consecutive line steps

$G' = \frac{A_k(n+1)}{A_k(n)} = \frac{A_k(n)}{A_k(n-1)}$

Then $G'^2 + 2i\lambda \sin(k\Delta x) - 1 = 0$

and

$G''_1 = \mu_1 = -i\lambda \sin(k\Delta x) + \sqrt{1 - \lambda^2 \sin^2(k\Delta x)}$

$G''_2 = \mu_2 = -i\lambda \sin(k\Delta x) - \sqrt{1 - \lambda^2 \sin^2(k\Delta x)}$

(As $\Delta t \to 0$, $\mu_1 \to 1$ and $\mu_2 \to -1$)

Its origin lies in the fact that the solution of the FD are independent between odd and even numbers These two solutions will evolve differently unless (a) the first time step generating $A_k(1)$ is excluded such that $\mid \beta \mid << \mid \alpha \mid$ and (b)any component of the solution $\|$ to $\vec{x_2}$ that might arise due to sound-off errors is periodically reduced****.

a)

$A_k(o)[1 - ip] = \alpha[-ip + \sqrt{1 - p^2}] + \beta[-ip - \sqrt{1 - p}]$

$A_k(o) = \alpha + \beta$

or

$A_k(o)(1 - ip) = A_k(o)[-ip + \sqrt{1 - p^2}] + \beta[-2\sqrt{1 - p^2}]$

$\beta_1 = A_k(\bullet)\frac{1 - \sqrt{1 - p^2}}{-2\sqrt{1 - p^2}} = A_k(o)[\frac{\lambda}{2} - \frac{1}{2}\frac{1}{\sqrt{1 - p^2}}]$

b)

$A_k(o) = \alpha[-ip + \sqrt{1 - p^2}] + \beta[-ip - \sqrt{1 - p^2}]$

$A_k(o) = \alpha + \beta$

$\beta_2 = A_k(o)\frac{1 + ip * -\sqrt{1 - p^2}}{-2\sqrt{1 - p^2}} = A_k(o)[\frac{\lambda}{2} - \frac{1 + ip}{2\sqrt{1 - p^2}}]$

The two eigemvectors of the leap-grog scheme are

$\vec{x_1} =$

$\vec{x_2} =$

$= \alpha\vec{x_1} + \beta\vec{x_2} p = \lambda \sin(k\pi\Delta x)$

We are looking for the method which produces the smaller computational mode (smallest β) from starting

a) $U_j^1 = U_j^0 - \frac{\lambda}{2}(U_{j+1}^0 - U_{j-1}^0)$

b) $U_j^1 = U_j^0$**

or in function of the test function

a) $A_k(1) = A_k(0) - \frac{\lambda}{2} A_k(0)(2i\sin(k\pi\Delta x)$

b) $A_k(1) = A_k(0)$

In order to have a stable scheme*, $\lambda \leq 1$ then $1 - p^2 \geq 0$ and we can write

$$\frac{|\beta_2|^2}{|\beta_1|^2} = \frac{(1-\sqrt{1-p^2}^2 + p^2}{(1-\sqrt{1-p^2}^2} \geq 1$$

Then a leapfrog integration of the advection equation $\frac{\partial \mu}{\partial t} = -x\frac{\partial \mu}{\partial x}$ should be started with a single forward step,

i.e. $U_j^1 = U_j^0 - \frac{\lambda}{2}(U_{j+1}^0 - U_{j-1}^0)$

(Smallest computational mode)

b) Upstream differencing (Donor all)

(7) $U_f^{n+1} = U_f^n - \lambda$

$U_{f+1}^n - U_j^n$ if $\lambda < 0$

$U_f^n - U_{f-1}^n$ if $\lambda > 0$

The amplification factor G is then equal to $1 - \lambda$

$e^{ik\Delta x} - 1$ if $\lambda < 0$

$1 - e^{-ik\Delta x}$ if $\lambda > 0$

This scheme is stable if $|\lambda| < 1$. Easy to implement, but not recommended because introduces* artificial dissipation. (computational verceility**)

(7) can be rewritten as

$U_f^{n+1} = U_f^n - \frac{\lambda}{2}(U_{f+1}^n - U_{f-1}^n) \leftarrow$ FTCS (unstable)

$+\frac{|\lambda|}{2}(\underline{U_{f+1}^n + U_{f-1}^n} - 2U_f^n) \leftarrow$ second order approximation to a definite equation.

$\alpha = \frac{c\Delta x}{2}$ represents the value for the eddy viscosity!

c) Diffusion scheme** (Also called Friedrich's scheme)
Already presented before

$$U_f^{n+1} = \frac{1}{2})(U_{f+1}^n + U_{f_1}^n) - \frac{\lambda}{2}(U_{f+1}^n - U_{f-1}^n)$$

The amplification factor is $G = \cos(k\Delta x) - i\lambda \sin(\Delta x)$
Stable if $\mid \lambda \mid< 1$

this scheme does introduce diffusion* to stabilize the FTCS scheme.

FTCS$\leftarrow$ $\underline{U_j^{n+1} = U_n^f - \frac{\lambda}{2}(U_{f+1}^n - U_{j-1}^n)}$+$\underline{\frac{1}{2}(U_{f+1}^n - U_{f-1}^n - 2U_f^n)}$ $\rightarrow$ diffusion

with $\underline{\alpha' = \frac{\Delta x^2}{2\Delta t}}$ $\leftarrow$Analytical viscosity

d) The Lax-Wendroff scheme

We saw that the various* ruco schemes did introduce artificial viscosity. The reasoning behind the Lax- Wendroff scheme is the following: can we stabilize he FTCD shown by adding the minimal amount of artificial damping? We can write the FD equation as:

$$U_j^{n+1} = U_n^f - \frac{\lambda}{2}(U_{f+1}^n - U_{g-1}^n) + \nu(U_{f-1}^n + U_{f+1}^n - 2U_j^n$$

$$V = 0 \qquad \nu = \frac{1}{2} \qquad \nu* = \frac{\mid \lambda \mid}{2}$$

The amplification factor for such a scheme is

$$(8) \quad \boxed{G = \frac{A_k^{n+1}}{A_k^n} = 1 - i\lambda \sin(k\Delta x) + 2\nu(\cos(k\Delta x) - 1)}$$

Stability is assured if $\mid G \mid \leq 1 + o(\Delta t)$ for all k
(9) $\mid G \mid^2 = 1 + (2\lambda^2 - 4\nu)[1 - \cos(k\Delta x)] + (4\nu^2 - \lambda^2)[1 - cos(k\Delta x)]^2$
$= 1 + (2\lambda^2 - 4\nu)p + (4\nu^2\lambda)p^2$.

For $\nu = \frac{\mid \lambda \mid}{2}$, G is a linear function of $p = 1 - \cos(k\Delta x)$ scheme strartle, upstream diff scheme.

We look for a better scheme, namely
$\nu < \frac{\mid \lambda \mid}{2}$ or $\lambda^2 > 4\nu^2$

(9) is maximum for $\boxed{p_{max} = \frac{\lambda^2 - 2\nu}{\lambda^2 - 4\nu^2}}$

and $\mid G \mid^2$ is then equal to at this point:

$$\boxed{1+\frac{(\lambda^2-2\nu)^2}{\lambda^2-4\nu^2}}$$

(10) then $\boxed{\max \mid G \mid^2 \leq 1+\frac{(\lambda^2-2\nu)^2}{\lambda^2-4\nu^2}, 0 \leq p \leq 2}$ (maximum not necessarily located in [0,2]**)

This implies $\lambda^2-2\nu)^2=0$ or $\boxed{\nu=\frac{\lambda^2}{2}}$

Further reduction is NOT possible.

$\nu < \frac{\lambda^2}{2}$

Max $\mid G \mid^2 = \Big|$ 1+ $(\lambda^2-2\nu)^2 \overline{{}_{\lambda^2-4\nu^2} \quad {}_{>1forp_max<2}}$