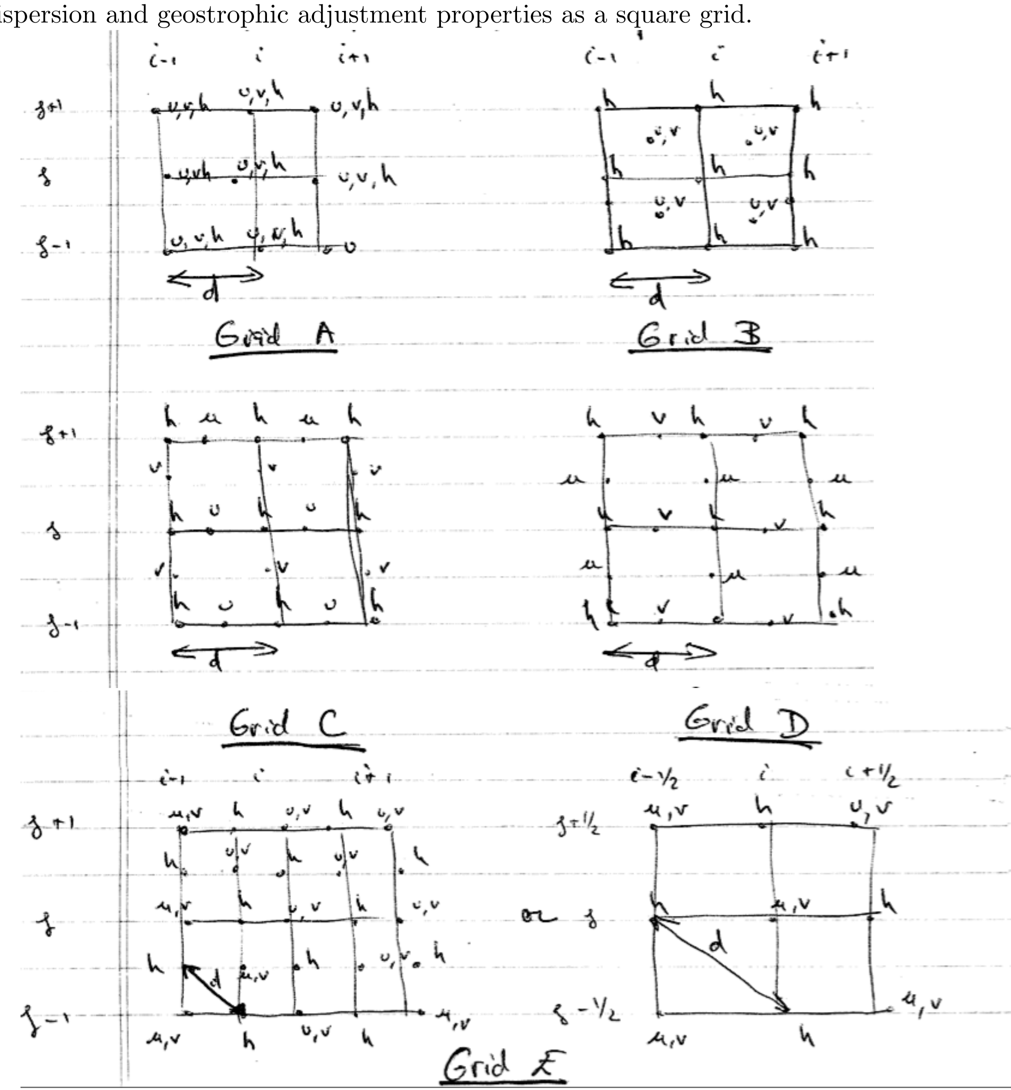
8. $\boxed{\text{Miscellaneous}}$

8.1 Grids

As shown by Homework 3, it can be to our advantage to use staggered grids in space.

Anakawa and Laub (1977) composed among others (Winmghogg, (1968); Schuestrad (1978)) five different arrangements of the dependent valuables for dispersion and geostrophic adjustment properties as a square grid.



We now, for purpose of the comparison, redefine the δ and ( ).

$(\delta_x \alpha) = \alpha_{i+\frac{1}{2},f} - \alpha_{i-\frac{1}{2},f}$

$(\vec{\alpha}^x)_{i,f} = \dfrac{1}{2}\alpha_{i+1}$

Same for the y direction.

For the shallow-water equations, we can rewrite the basic linear equations as:

Grid A:

$$\frac{\partial \mu}{\partial t} - fv + \frac{g}{d}(\vec{\delta_x h})^x = 0$$
$$\frac{\partial v}{\partial t} + f\mu + (\frac{g}{d})(\vec{\delta_y h})^y = 0$$
$$\frac{\partial h}{\partial t} + (\frac{H}{d})[(\vec{\delta_x \mu})^x + (\vec{\delta_y v})^y] = 0$$

Grid B:

$$\frac{\partial \mu}{\partial t} - fv + (\frac{g}{d})(\vec{\delta_x h})^y = 0$$
$$\frac{\partial v}{\partial t} + f\mu + (\frac{g}{d})(\vec{\delta_y h})^x = 0$$
$$\frac{\partial h}{\partial t} + (\frac{H}{d})[(\vec{\delta_x \mu})^y + (\vec{\delta_y v})^x] = 0$$

Grid C:

$$\frac{\partial \mu}{\partial t} - f\vec{v}^{xy} + (\frac{g}{d})(\delta_x h) = 0$$
$$\frac{\partial v}{\partial t} + f\vec{\mu}^{xy} + (\frac{g}{d})(\delta_y h) = 0$$
$$\frac{\partial h}{\partial t} + (\frac{H}{d})[(\delta_x \mu) + (\delta_y v)] = 0$$

Grid D:

$$\frac{\partial \mu}{\partial t} - f\vec{v}^{xy} + (\frac{g}{d})(\vec{\delta_x h})^{xy} = 0$$
$$\frac{\partial v}{\partial t} + f\vec{\mu}^{xy} + (\frac{g}{d})(\vec{\delta_y h})^{xy} = 0$$
$$\frac{\partial h}{\partial t} + (\frac{H}{d}[(\vec{\delta_x \mu})^{xy} + (\vec{\delta_y v})^{xy}] = 0$$

Grid E

$$\frac{\partial \mu}{\partial t} - fv + (\frac{g}{d*})(\delta_x h) = 0$$
$$\frac{\partial v}{\partial t} + f\mu + (\frac{g}{d*})(\delta_y h) = 0$$
$$\frac{\partial h}{\partial t} + (\frac{H}{d*})[(\delta_x \mu) + (\delta_y v)] = 0$$

In the latter (Grid E), a grid destrae* of $\sqrt{2d} = d*$ gives the same number of grid points as the other schemes given a two-dimensional domain.

In order to illustrate the properties of these five schemes, we consider the one dimensional linear equations:

$$\frac{\partial \mu}{\partial t} - fv + g(\frac{\partial h}{\partial x}) = 0$$
$$\frac{\partial v}{\partial t} + f\mu = 0$$
$$\frac{\partial h}{\partial t} + H(\frac{\partial \mu}{\partial x}) = 0$$

Eliminating v, h yields to:

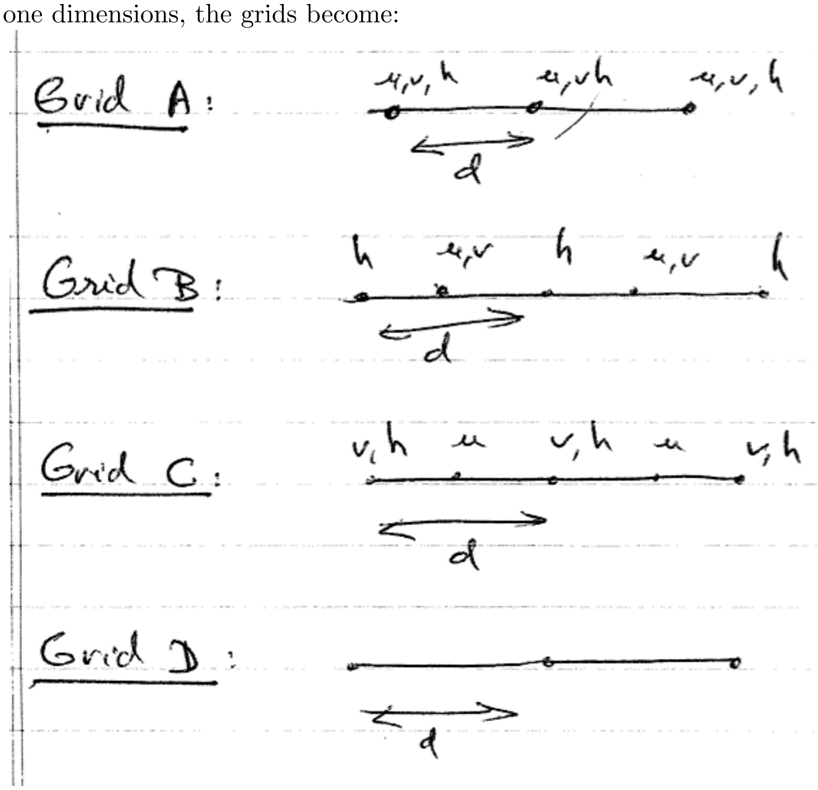
$$(1)\ \frac{\partial^2 \mu}{\partial t^2} + f^2 \mu - gH(\frac{\partial^2 \mu}{\partial x^2}) = 0$$

If we assume the solution to be proportional to $e^{i(kx-\omega t)}$, then the angular frequency ω is given by:

$$(\frac{\omega}{f})^2 = 1 + gH(\frac{k}{f})^2 = 1 + k^2 \underline{Rd^2}$$

(Inertia-Gravity waves) Extend Rossby Radians of Deformation* $\frac{\sqrt{gH}}{f}$

We can now examine the effect of the space discretization at the frequency. In one dimensions, the grids become:



Grid E is equivalent to A, but with a smaller grid size. For the different schemes the following frequencies are obtained (Anakawa an Laub, 1977)

Grid A:

$$(\frac{\omega}{f})^2 = 1 + (\frac{Rd}{d})^2 \sin^2(kd)$$

Grid B:

$$(\frac{\omega}{f})^2 = 1 + 4(\frac{Rd}{d})^2 \sin^2(\frac{kd}{2})$$

Grid C:

$$(\frac{\omega}{f})^2 = \cos^2(\frac{kd}{2}) + 4(\frac{Rd}{d})^2 \sin^2(\frac{kd}{2})$$

Grid D:

$$(\frac{\omega}{f})^2 = \cos^2(\frac{kd}{2}) + (\frac{Rd}{d})^2 \sin^2(kd)$$

For all cases, $\frac{\omega}{f}$ depends on a kd and $\frac{Rd}{d}$. The wavelength of the shorter resolvable wave is 2d (corresponding wave when $k_m ax = \frac{\pi}{d}$) We therefore exercise* the range* $o < kd < \pi$.

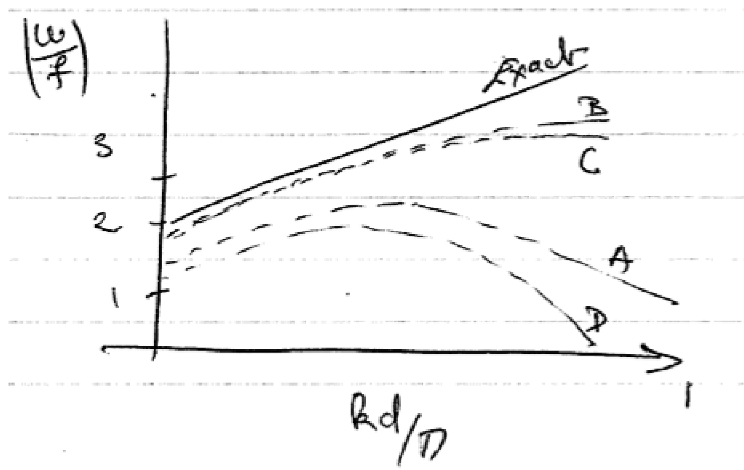
Grid A: Maximum for $kd = \frac{\pi}{2}$ with a sevc* group velocity $\frac{\partial\omega}{\partial k}$

Grid B: For near zero Rd, modiscially* increasing.

Grid C: Monotonically increasing for $\frac{Rd}{d} > \frac{1}{2}$ and ———— decreasing for $\frac{Rd}{d} < \frac{1}{2}$. For $\frac{Rd}{d} = \frac{1}{2}, \omega^2 f^2$ and the group velocity is zero for all k.

Grid D: ω reaches a maxium for $fracRdd^2 \cos(kd) = \frac{1}{4}$.
$kd = \pi$ is a stationary wave.

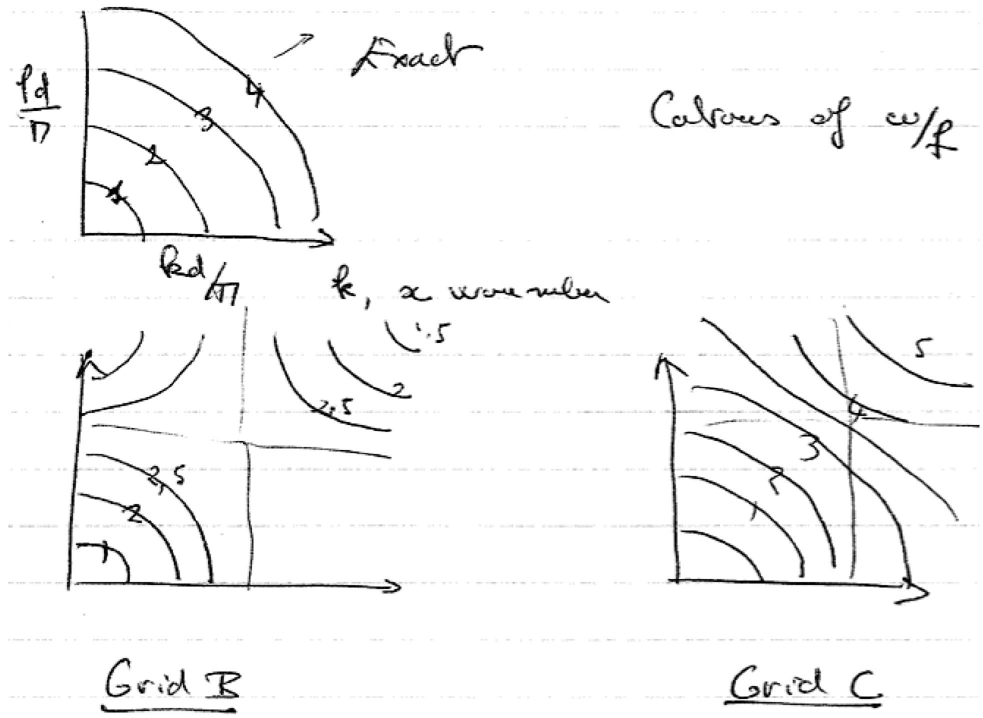
For this one dimensional case, grid B is the most satisfying. However for $\frac{Rd}{d}$ larger than $\frac{1}{2}$, grid C is as good as grid B.



$$\boxed{\frac{Rd}{d} = 2}$$

As shown in the homework, staggered grids require half the time step that is required for unstaggered grids. However, the eckia* computer time is worth with B and C since they give better structure for the shorter waves. Inipertait* after a geostrophic adjustment.

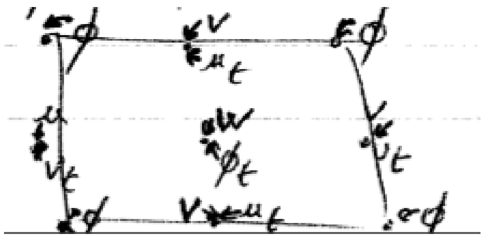
In the two dimensional case, there is a difficulty with B as shown by Anakawa and Laub (1977).



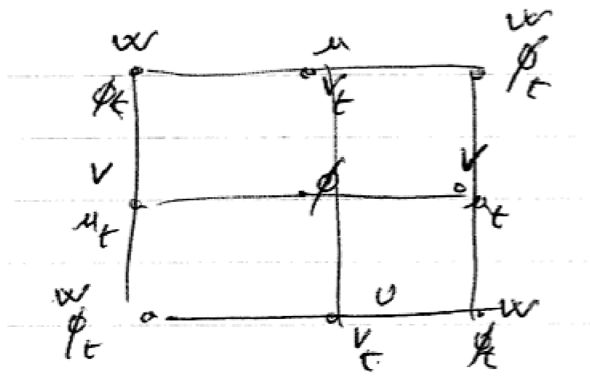
For $\frac{R_d}{d} > 1$, Grid C better

$\frac{R_d}{d} < 1$, Grid B better

The grids can also be staggered in time as proposed by Eliassen (1956) for a baroclinic * PE model. Has escalled* adjustment properties:



Even time steps



Odd time steps

An analysis of this scheme can be found by (Anakawa and Mesiyer (1976))

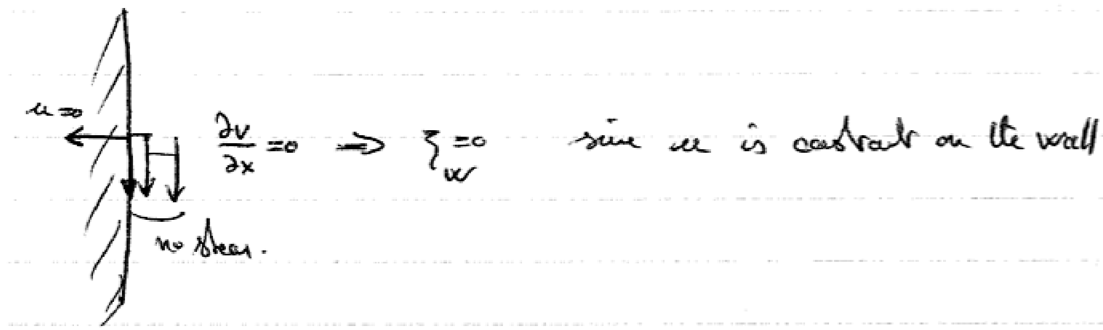
8.2 Boundary Conditions

They are important since they often define the problem. A first order ordinary differential equation such as $\frac{df}{dx} = 0$ specifies the solution as a constant. The boundary condition determines the value of this constant. A first order partial differential equation such as $\frac{\partial f(x, y,)}{\partial x} = 0$ specifies very little of the solution. Any function $g(y)$ satisfies the equation and the boundary condition must specify $g(y)$.

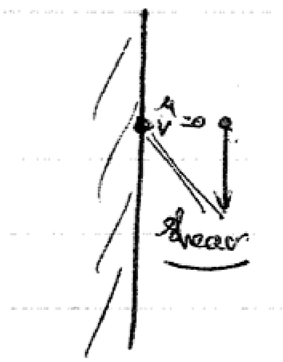
The specification of computational boundary conditions, besides coeffecting numerical stability , greatly affects the accuracy of the of the finite difference solution.

Boundary conditions at the walls.

In addition to the no-normal flux through the wall (normal velocity is zero), viscous* boundary conditions are defined for the integration. The two mesh contours* are free-slip (the normal derivative of the velocity $\|$ to the wall is equal to zero. $\leftrightarrow$ velocity equal to zero at the boundary



or no-slip (both μ and r are equal to zero on the boundary, $\zeta_w \neq 0$



Implementation of this boundary condition various depending on the mesh used.

Let's consider the vorticity equation and Possion equation. We then need a boundary condition for the vorticity equation and one for the Poisson equation.

For free slip, in a regular mesh, $\zeta = 0$ and $\psi = 0$ are the needed BCS.

For no-slip, the streamfunction ψ can be expanded in a Taylor series (vertical* wall)

(3) $\psi_{i+1,f} = \psi_{i,f} + \frac{\partial \psi}{\partial x} \mid_{i,f} \Delta x + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} \mid_{i,f} \Delta x^2 + ...$

The velocity $v = 0$ by the no-slip condition and $\frac{\partial \psi}{\partial x}_{i,f} = 0$ and $\frac{\partial^2 \psi}{\partial x^2} \mid_{i,f} = \frac{\partial u}{\partial x} \mid_{i,f}$

$\zeta = \frac{\partial u}{\partial x} - \frac{\partial \mu}{\partial y} \rightarrow \frac{\partial \mu}{\partial y} = 0$ because of $\mu = 0$ and construct* along the wall.

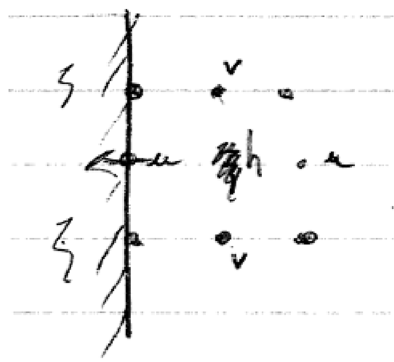
then $\zeta_w = \frac{\partial u}{\partial x}_{i,j}$

From (3), we derive the expression for the vorticity at the wall.

$$\zeta_w = \frac{2(\psi_{i+1,j} - \psi_{i,f})}{\Delta x^2}$$

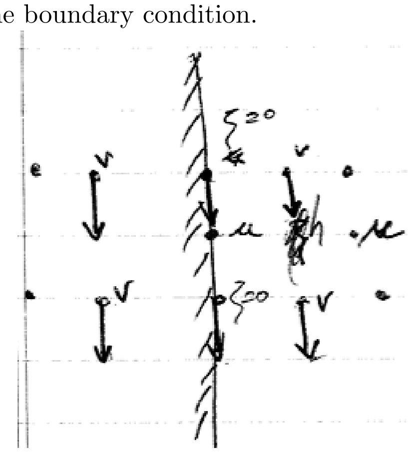
One has to be careful about the possible oversimplification of boundary conditions. For a no-slip surface wall, $\mu = v = 0$ and $\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0$. Either of the conditions are sufficient, but we cannot use both to solve the Poisson equation. On the other hand, $\psi_w = 0$ is used for the Poisson equation and the $\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0$ for the vorticity equations.

In staggered grid systems, the expression for the vorticity might be different. Let's consider the case of the C-grid for a primitive equation model.



The μ are defined on the wall, but note the v -velocities or the vorticity.

We therefore need to specify values of v "in" the boundary to recover the the boundary condition.



Free-Slip

The V inside the wall has the same values as the one outside $\rightarrow \zeta_w = 0$.

<u>No-Slip</u>

The V under* the wall are opposite such that $v = 0$ at the wall, $\zeta = \dfrac{2v}{\Delta x} = \dfrac{V_{out} - V_{in}}{\Delta x}$

b) <u>Upper-boundary</u> (Rigid lid approximation)

In the case of a primative equation model, the time-step must satisfy $\Delta t < \dfrac{\Delta x}{c_{max}}$ where c_{max} is the maximum phase speed that can occur in the system. These equations permit external and internal gravity waves, inertial osallations and Rossby waves. The external gravity waves have the speed $cv\sqrt{gH}$ which gives $c = 200m/s$ for $H = 400m \rightarrow$ very small Δt capered * to the typical time scale of oceanic metrics.

We can exclude the gravity waves by placving a lid at the surface or setting $W = 0$ at the upper boundary. A longer time step can be taken since the internal waves have a $c \leq 10m/s$

<u>Basic Equations:</u>

(4) $\dfrac{d\vec{v}}{dt} = -\dfrac{1}{p_0}\vec{\nabla}p - f\vec{k}x\vec{v} + \vec{F}$

(5) $\dfrac{dp}{dz} = -pg$

(6) $\dfrac{dw}{dz} + \nabla \cdot \vec{v} = 0$

Flat bottom + ridged lid give $w(0) = w(-H) = 0$. Integration of the continuity equation (6) gives

$\boxed{\nabla \cdot < \vec{v} >= 0}$ (7)

Where $<(\ )>= H^{-1}\int_{-H}^{0}(\ )dz$

The vertically averaged $-H$ component* is non divergent and a streamfunction can be introduced such that:

$< \vec{v} >= \vec{k} \times \vec{\nabla}\psi$ (8)

We now take the vertical average of the equation of notation (4)

$$\frac{\partial}{\partial t}\nabla^2\psi = -\vec{k} \cdot \vec{\nabla}x < \vec{v} \cdot \nabla\vec{v} + w\frac{\partial\vec{v}}{\partial z} + F > - < v > \cdot\nabla f \ (10)$$

This equation can be used to predict the mean vertical current** ψ and $< v >$

The departure from the mean is defined as

$v =< v > +v'$ (11)

Substituting in (4) and subtracting (9), we obtain an equation for v'

$$\frac{\partial\vec{v'}}{\partial t} = -(\vec{v} \cdot \nabla\vec{v} + w\frac{\partial\vec{v}}{\partial z}) + < \vec{v} \cdot \nabla\vec{v} + w\frac{\partial\vec{v}}{\partial z} > -f\vec{k}x\vec{v'} - \frac{1}{P_0}\vec{\nabla}p' + F - < F >$$

where $p' = p- < p >$

The hydrostatic equation (5) becomes

$\frac{\partial p'}{\partial z} = -gp$ which integrated from $-Htoz$ gives

(13) $p' - p'(-H) - g\int_{-H}^{z} pdz$

$< p' >= 0 \rightarrow$ (13) can be rewritten as

(14) $p' = -g\int_{-H}^{z} pdz + g < \int_{-H}^{z} pdz >$

* First step, calculation of ψ using (10). This requires a Poisson solver. ψ can b used to calculate $< \vec{v} >$ from (8).

We then solve (12) for v' and the thermodynamic equation is advareed *

(15) $\frac{d\theta}{dt} = A_\theta\nabla^2\theta + k_\theta\frac{d^2\theta}{dz^2} + r_c(\theta)$

and $p = p_o[1 - \alpha\theta]$, new density field.

p' is then computed* for use in (12).

Integration of the continuity equation gives the vertical velocity.

Another approach for large scale modeling other than the rigid lid is the use of *split-explicit* time differencing scheme which treat the baraotropic* and baroclinic* component of the flow with two different time step (small for baratropic (9), fast for baroclinic (12)).

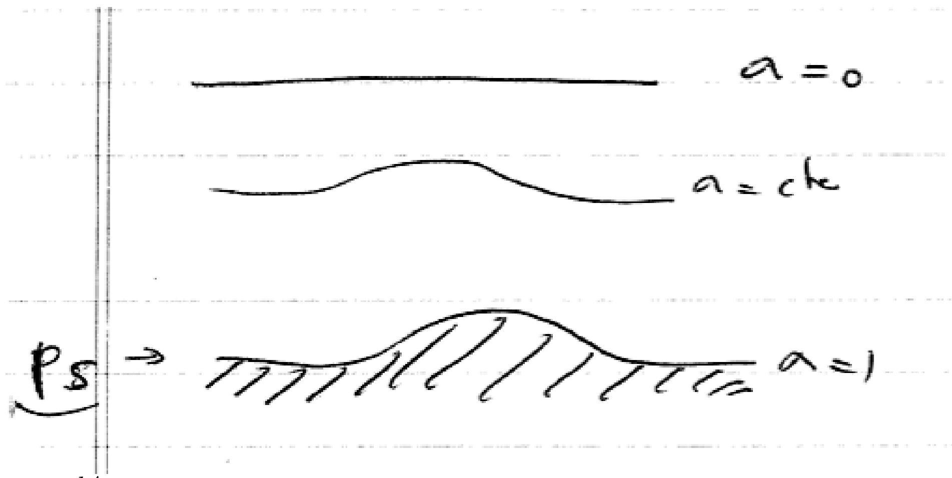
This method treats the traveling Rossby waves more properly and in the case of irregular topography, avoids the use of a fast Poisson solver (problematic and a non-rectangular domain).

Bolton topography

Numerical techniques dealing with the modeling of topography in oceanic and atmospheric models depend on the assumed vertical coordinate. They *bring* many difficulties regarding the model stability or the stability of the result.

In a z-coordiniate, the most natural choice, each level is immersion is independent of the bolton topography as well as of the horizontal location. The physical variables of the system are set equal to zero at the grid-points located inside the boltan topography. The boltan topography is then represented by a succession of steps. Such a coordinate may be adapted for steep reliefs, but not for gentle slopes.

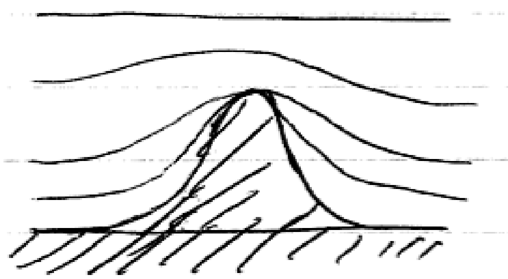
An alternative, extremely popular in the atmospheric models, is the α - coordinates (or normalized pressure coordinate) after Phillips (1937) with $\alpha = \frac{p}{p_s}$ where p_s is the pressure at the bottom and is a function of x, y, t



with $\frac{dA}{dt} = 0$ at the surface.

It conveniently avoids uncomfortable problems with the lowest boundary conditions, nevertheless, this coordinate also sets many problems, especially the non-cancellation of truncation errors(Smagoriusky, et al. 1967). (See Maltier and Willams for a review of the α - coordinate system). Applications to ocean models is fairly recent and preliminary conclusions are that α - coordinates does pretty well with gentle slope (not steep) and vise-versa for z-coordinates.

The other coordinate which seems to handle all topographic situations well is the isopycnic system. Major problem is the intersection of the layers with the topography. This can be handled with a scheme which conserves the posc* of the layer thickness such as FCT or Smeltcheuic* Preliminary results from here (UM) are extremely primary*. (Bleck and Smith, 1990)



Steep topography in ocean models is under investigation (ART, ONR).

d) Irregular Boundaries

Irregular Boundaries are everywhere in the ocean and do not facilitate the task of integrating the equations.

For example, in the case of a primitive equation model, introduction of a $(x, y)\nu$ to p–* does not feuite* anymore to solve directly the Poisson equation and an iterative method such as the SOR (Successive Over - Relaxation techniques) must be used or the split-*explicit* approach described in the previous section. The latter allows also for irregular lateral boundary conditions.

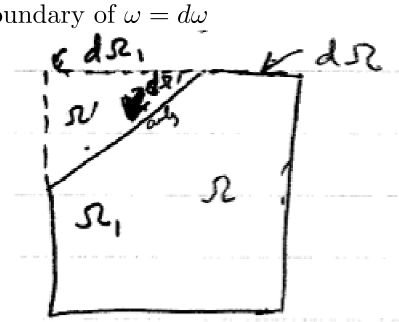
There is one method for either PE-flat bottom or QG which still allows for the use of a direct solver : The Capacitive Matrix Method (Carmuis and Mysak, 1988 OPO adopted from Hockney, 1970)

The capacitive matrix method is a technique for *extracting* the usefulness of direct solvers to now rectangular domains.The major computational borders* of the method is that it requires to call twice the Poisson solver. It is in general more efficient then using an iterative method.

We obtain a field which satisfies:

(16) $\nabla^2\psi = \zeta$

ψ, ζ functions of x, y in a domain ω with the condition $\psi = \psi_b$ on the boundary of $\omega = d\omega$



We can embed this domain into a rectangular area ω_1. The difference is ω' with $d\omega_1$ and $d\omega'$ their respective boundaries.

We first obtain though a direct solver the field ψ_1 in the ω_1 domain by solving

$\nabla^2\psi_1 = \zeta_1$
when: $\zeta_1 = \zeta$ in ω, ζ_1 = arbitrary in ω' and can be taken to be equal to zero.

Formally, the solution can be written as a function of the Green's* functions associated with the operator ∇^2 such that

$$\psi_1 = \iint_{\omega_1} G(x, y, x', y')\zeta_1(x', y')dx'dy' + \int_{d\omega_1} \psi_1 \frac{\partial G}{\partial n} dp'$$

The Green's function G satifies with δ = Dirac* function.

$\nabla^2 G = \delta(x - x')\delta(y - y')$ on ω_1 $G = 0$ on $d\omega_1$

The essence of the capacitive matrix method is now to modify is now to modify ψ_1 such that $\psi_1 - \psi_b$ and $d\omega'$. We then modify ζ_1 and $d\omega'$ such that the solution satisfies the boundary condition.

We denoted $\theta(\vec{s})$, a function valid on an $\delta\omega'$ only $\vec{s}$ vector on the boundary of $\delta\omega'$). We consider the function $\mu(x, y)$ such that

(20) $\nabla^2\mu = \zeta_1 + \theta$ in ω_1 with $\mu = \psi_b$ on $d\omega_1$

If now $\theta(\vec{s})$ is chosen such that μ satifies $\mu = \psi_b$ and $\delta\omega'$ then $\psi(x, y) = \mu$ in ω and the solution is found.

To determine $\theta(\vec{s})$, we use (18) and (19) to get the solution of (20).

(21) $\mu(x, y) = \phi_1 + \int_{\delta\omega'} \theta(\vec{s})G(x, y, \vec{s})d\vec{s'}$

$\mu = \psi_b on \delta\omega' \rightarrow$ (21) reduces to

(22) $\psi_b = \psi_1(\vec{s}) + \int_{\delta\omega'} \theta(\vec{s'})G(x, y, \vec{s'}d\vec{s'}$

which is an integral equation that determines $\theta(\vec{s})$.

Application to finite-difference:

There are $\delta\omega'$ passes through a set of grid points refered to as the irregular boundary points.

1) The numerical algorithm first requires that we apply a direct solver to obtain ψ_1 in the rectangle* given the forcing* ζ_1

2) Next the modifying function θ is obtained from the values of ψ_1 along $\delta\omega'$ by using (22)(discredited version).

3) The direct solver is employed a second time to solve (20) and to obtain ψ.

(22) is rewritten as $\psi_b = \psi_1 + \Delta s^2 G\theta$ where Δs = grid interval, θ_1, ψ_b, ψ_1 are column vectors of length m, (m = number of irregular boundary points)

The finite difference Green function, G, is a matrix M $\times$ m

$\theta = \Delta s^{-2} C(\psi_b - \psi_1)$
where $C = G^{-1}$ = capacitance* matrix
C has to be determined only once.

G is determined from (14) by defining a delta function of strength Δs^{-2} and is placed at one irregular boundary. Must satisfy for all the points.

Example: