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Computational Design for Long-Term Numerical Integration of the Equations of Fluid Motion: Two-Dimensional Incompressible Flow. Part I[1]

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The integral constraints on quadratic quantities of physical importance, such as conservation of mean kinetic energy and mean square vorticity, will not be maintained in finite difference analogues of the equation of motion for two-dimensional incompressible flow, unless the finite difference Jacobian expression for the advection term is restricted to a form which properly represents the interaction between grid points, as derived in this paper. It is shown that the derived form of the finite difference Jacobian prevents nonlinear computational instability and thereby permits long-term numerical integrations. © 1966 Academic Press

INTRODUCTION

A major difficulty, which has blocked progress in long-term numerical integration of the equations of fluid motion, has been nonlinear computational instability of the finite difference analogues of the governing differential equations. The existence and cause of this instability was first called to our attention by Phillips [1, 2].

The instability can be illustrated by integration of the vorticity equation for two-dimensional incompressible flow,

$$\frac{\partial \zeta}{\partial t} + \mathbf{v} \cdot \nabla \zeta = 0, \tag{1}$$

where

$$\mathbf{v} = \mathbf{k} \times \nabla \psi,$$
$$\zeta = \mathbf{k} \cdot \nabla \times \mathbf{v} \equiv \nabla^2 \psi,$$

and ψ is the stream function, ∇ is the two-dimensional del operator, and $\mathbf{k}$ is unit vector normal to the plane of motion.

Equation (1) can be rewritten as

$$\frac{\partial \zeta}{\partial t} = J(\zeta, \psi), \tag{2}$$

or

$$\frac{\partial\, \nabla^2 \psi}{\partial t} = J(\nabla^2 \psi, \psi), \tag{3}$$

where J is the Jacobian operator with respect to the rectangular coordinates, x and y, in the plane.

When the Jacobian in this equation is replaced by space-differences of the usual form,

$$\begin{aligned}\mathbb{J}_{i,j}(\zeta, \psi) = \frac{1}{4d^2}[&(\zeta_{i+1,j} - \zeta_{i-1,j})(\psi_{i,j+1} - \psi_{i,j-1}) \\ &- (\zeta_{i,j+1} - \zeta_{i,j-1})(\psi_{i+1,j} - \psi_{i-1,j})],\end{aligned} \tag{4}$$

where i is the finite-difference grid index in x, j is the index in y, and d is the grid interval, and the equation is integrated over some tens of time steps, using an ordinary time-centered differencing scheme, it is found that the solution begins to show a characteristic structure termed "stretching" or "noodling" [3, 4]. This is a structure in which the motion degenerates into eddies of a few grid intervals in size and of elongated, filamented shape.

The early stages of this noodling can be due to physical processes, but, once formed, the eddies usually intensify without limit, causing computational instability and explosive growth of the total kinetic energy of the system. It is also observed that as integration proceeds the energy is distributed over a broader and broader range of wave number.

Platzman [3] recognized the existence of "aliasing errors," or errors due to misrepresentation of the shorter waves because of the inability of the finite grid to properly resolve them. Phillips [2] further showed that the above computational instability can be caused by this "aliasing." In addition, Richtmeyer [5] pointed out that, in a one-dimensional hyperbolic problem, if the disturbance is out of the properly defined linear range the rate of false growth

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cannot be reduced by shortening the time interval. Miyakoda [6] showed that this type of computational instability (which Phillips found for the nonlinear equations) can also occur in a linear equation with nonconstant coefficients.

For long-term integration of the equations of fluid motion it is necessary to overcome the computational instability through proper computational design of the integration. Because this nonlinear instability has its origin in space-truncation errors, this paper will be concerned with the proper form of space-differencing. It will describe the principle and give some examples of space-difference schemes in which the nonlinear computational instability does not appear. The paper will discuss, moreover, not only the stability of the difference scheme, but also how well the scheme similates other important properties of the continuous fluid, such as the constraint on the spectral distribution of its energy.

I. CONSTRAINTS ON THE ADVECTION TERM

Equation (1) implies the conservation of vorticity for individual fluid particles and, therefore, the frequency distribution of the vorticities of the fluid elements does not change with time in two-dimensional incompressible flow. Moreover, since the advection of vorticity, like the advection of any quantity in two-dimensional incompressible flow, can be expressed by a Jacobian, as in (2) or (3), we can easily see that there are strong integral constraints on the advection term, which come from the nature of the Jacobian. Among these constraints, the following are the simplest ones with which we are concerned:

$$\overline{J(p,q)} = 0, \tag{5}$$

$$\overline{pJ(p,q)} = 0, \tag{6}$$

$$\overline{qJ(p,q)} = 0, \tag{7}$$

where p and q are any arguments and the bar denotes the average over the domain in the plane of motion, along the boundary of which either p or q is constant. From these integral constraints, applied to the advection of vorticity, we can see that the mean vorticity, $\overline{\zeta}$, the mean kinetic energy, $K \equiv \frac{1}{2}\overline{\mathbf{v}^2} = \frac{1}{2}\overline{(\nabla\psi)^2}$, and the mean square vorticity, $2V \equiv \overline{\zeta^2} = \overline{(\nabla^2\psi)^2}$, in a closed domain, across the boundary of which there is no inflow or outflow, are conserved with time.

Expanding ψ into the series of orthogonal harmonic functions, ψ_n, which satisfy

$$\nabla^2\psi_n + k_n^2\psi_n = 0, \tag{8}$$

we get

$$\frac{dK}{dt} = \frac{d}{dt}\sum_n K_n = 0, \quad K_n \equiv \frac{1}{2}\overline{(\nabla\psi_n)^2}, \tag{9}$$

and

$$\frac{dV}{dt} = \frac{d}{dt}\sum_n V_n = 0, \quad V_n \equiv \frac{1}{2}\overline{(\nabla^2\psi_n)^2} = k_n^2 K_n. \tag{10}$$

Therefore, the average wave number, k, defined by

$$k^2 \equiv \frac{\sum_n k_n^2 K_n}{\sum_n K_n}, \tag{11}$$

is conserved with time. This shows that no systematic one-way cascade of energy into shorter waves can occur in two-dimensional incompressible flow, as Fjørtoft [7] pointed out. If we consider, for example, three waves (or three groups of waves, each of which has a characteristic average scale), only the following energy exchanges are possible:

$$K_L \leftarrow K_M \rightarrow K_S,$$

or

$$K_L \rightarrow K_M \leftarrow K_S,$$

where K_L, K_M, and K_S are the mean kinetic energies of the long wave(s), medium wave(s), and short wave(s), respectively. Moreover, it turns out that relatively little energy exchange can take place between K_M and K_S, compared with the energy exchange between K_L and K_M.

It can also be shown, from the conservation of mean square vorticity, that the mean square total deformation, $\overline{D_1^2 + D_2^2}$, is also conserved. Here

$$\left.\begin{aligned} D_1 &\equiv \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right), \\ D_2 &\equiv \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \end{aligned}\right\} \tag{12}$$

$$u \equiv -\frac{\partial\psi}{\partial y}, \quad v \equiv \frac{\partial\psi}{\partial x}, \tag{13}$$

and there is an identity:

$$4J(u,v) = \zeta^2 - 4(D_1^2 + D_2^2). \tag{14}$$

As we have seen so far, the simple integral requirements, (5), (6), and (7), lead to important integral constraints on two-dimensional incompressible flow. But not only the mean vorticity and the quadratic means (such as mean kinetic energy, mean square vorticity, and mean magnitude of deformation) are constrained. The spectral energy distribution is also constrained, because the average wave number defined by (11) is conserved. Of course, these constraints are not sufficient to keep the frequency distribution of vorticity constant. However, it should be noted that the constraints on the mean vorticity, $\bar{\zeta}$, and on the mean square vorticity, $\overline{\zeta^2}$, are the constraints on the first and second moments of the frequency distribution of the vorticity.

One can easily visualize that in the usual scheme, given by (4), these integral constraints might not be maintained in a proper way. But if we can find a finite difference scheme which has constraints analogous to the integral constraints of the differential form, the solution will not show the false "noodling," followed by computational instability.

If we are only concerned with avoiding the computational instability, the conservation of either of the quadratic means (the mean kinetic energy or mean square vorticity) will be sufficient. But it is very desirable to require the conservation of both, because together they are a constraint on the spectral change of energy, as previously shown. Moreover, conservation of only one of these quadratic means is equivalent to the abandonment of the Jacobian property that $J(p, q) = -J(q, p)$, and hence that $J(p, p) = 0$.

It is known that the spectral computation of the Jacobian in wave number space, by means of truncated Fourier series (or spherical harmonics for the motion on a sphere), allows the conservation of the quadratic quantities. The energy and the square of the vorticity can be transferred from one wave to another, in a consistent manner, without the false gain or loss of these quantities. However, spectral computation has a practical disadvantage, in that the computation time increases as the square of the number of degrees of freedom, whereas there is only a linear increase of the computing time when using a finite difference scheme.

Our problem, then, is to find a finite difference scheme for the Jacobian, by means of which the two quadratic quantities, the kinetic energy and the square of the vorticity, are transferred, in a two-dimensional plane, from one grid point to another, without false gain or loss. In this way, the integral constraints on the quadratic quantities will be maintained when the integration is replaced by the summation of the quantities at the discrete grid points.

Lorenz [8], in dealing with the equations for a three-dimensional motion, showed how one can maintain integral constraints on quadratic quantities when vertical derivatives are replaced by vertical finite differences. In that work, he kept the horizontal derivatives in their differential form. But his procedure for the single vertical dimension gives us the clue to the treatment of the two-dimensional horizontal differencing.

II. FINITE DIFFERENCE ANALOGUES OF THE JACOBIAN

The finite difference analogue of the Jacobian at the grid point (i, j) may be written, in a relatively general form, as

$$\mathbb{J}_{i,j}(\zeta, \psi) = \sum_{i',j'} \sum_{i'',j''} c_{i,j;i',j';i'',j''}\, \zeta_{i+i',j+j'}\, \psi_{i+i'',j+j''}, \tag{15}$$

where $\zeta_{i+i',j+j'}$ is the vorticity at a neighboring grid point $(i + i', j + j')$ and $\psi_{i+i'',j+j''}$ is the stream function at a neighboring grid point $(i + i'', j + j'')$. The coefficients $c_{i,j;i',j';i'',j''}$ must be chosen in such a way that (15) is an approximation to the Jacobian with the order of accuracy we need. In addition, we have the requirements mentioned in the last section, which are now constraints on these coefficients.

In order to see when the square of the vorticity is conserved, it is convenient to define

$$a_{i,j;\,i+i',j+j'} \equiv \sum_{i'',j''} c_{i,j;\,i',j';\,i'',j''}\, \psi_{i+i'',j+j''}; \tag{16}$$

thus $a_{i,j;i+i',j+j'}$ is a linear combination of ψ, or, in fact, a linear combination of the velocity components as expressed by finite differences of the stream function. Then we have

$$\mathbb{J}_{i,j}(\zeta, \psi) = \sum_{i',j'} a_{i,j;\,i+i',j+j'}\, \zeta_{i+i',j+j'}. \tag{17}$$

When all of the $\zeta_{i+i',j+j'}$ are formally put equal to a constant, the Jacobian must vanish, regardless of the value of the constant. Thus we have

$$\sum_{i',j'} a_{i,j;\,i+i',j+j'} = 0, \tag{18}$$

which is a finite difference expression for $\nabla \cdot \mathbf{v} = 0$, as we shall see, later, more clearly.

Multiplying (15) by $2\zeta_{i,j}$, we obtain

$$2\zeta_{i,j}\mathbb{J}_{i,j}(\zeta, \psi) = \sum_{i',j'} 2a_{i,j;\,i+i',j+j'}\, \zeta_{i,j}\, \zeta_{i+i',j+j'}. \tag{19}$$

From (2), we see that the left hand side of (19) is the time change of $\zeta^2_{i,j}$ due to advection. Therefore, we can interpret the term $2a_{i,j;\,i+i',j+j'}\ \zeta_{i,j}\zeta_{i+i',j+j'}$ as the square vorticity gain at the grid point (i, j) due to the interaction with the grid point $(i + i', j + j')$. Similarly, $2a_{i+i',j+j';\,i,j}\ \zeta_{i+i',j+j'}\ \zeta_{i,j}$ can be interpreted as the square vorticity gain at the grid point

$(i + i', j + j')$ due to the interaction with the grid point (i, j). These two quantities must have the same magnitude and opposite sign, regardless of the values of $\zeta_{i,j}$ and $\zeta_{i+i',j+j'}$, in order to avoid false production of square vorticity. Therefore, we have the requirement

$$a_{i+i',j+j';\,i,j} = -a_{i,j;\,i+i',j+j'}, \tag{20}$$

in particular

$$a_{i,j;\,i,j} = 0, \tag{20$'$}$$

if the square vorticity is to be conserved in the finite difference scheme.

Replacing i by $i - i'$ and j by $j - j'$ in (20), we get

$$a_{i,j;\,i-i',j-j'} = -a_{i-i',j-j';\,i,j}. \tag{21}$$

Equations (17) and (18) are now rewritten as

$$\mathbb{J}_{i,j}(\zeta, \psi) = \sum_{i',j'}{}^{*} [a_{i,j;\,i+i',j+j'}\, \zeta_{i+i',j+j'} - a_{i-i',j-j';\,i,j}\, \zeta_{i-i',j-j'}], \tag{22}$$

$$\sum_{i',j'}{}^{*} [a_{i,j,i+i',j+j'} - a_{i-i',j-j';\,i,j}] = 0, \tag{23}$$

where $\sum_{i',j'}^{*}$ denotes the summation for the indices $j' > 0$, $i' \gtreqqless 0$ and $j' = 0$, $i' > 0$. Taking into account (23), (22) can also be rewritten as

$$\mathbb{J}_{i,j}(\zeta, \psi) = \sum_{i',j'}{}^{*} [a_{i,j;\,i+i',j+j'}\, (\zeta_{i+i',j+j'} - \zeta_{i,j}) + a_{i-i',j-j';\,i,j}\, (\zeta_{i,j} - \zeta_{i-i',j-j'})], \tag{24}$$

or

$$\mathbb{J}_{i,j}(\zeta, \psi) = \sum_{i',j'}{}^{*} [a_{i,j;\,i+i',j+j'}\, (\zeta_{i+i',j+j'} + \zeta_{i,j}) - a_{i-i',j-j';\,i,j}\, (\zeta_{i,j} + \zeta_{i-i',j-j'})]. \tag{25}$$

Equations (23), (24), and (25) correspond to the differential forms

$$-\tfrac{1}{2} \nabla \cdot \mathbf{v} = 0, \tag{26}$$

$$J(\zeta, \psi) = -\, \mathbf{v} \cdot \nabla \zeta, \tag{27}$$

and

$$J(\zeta, \psi) = -\, \nabla \cdot (\mathbf{v}\zeta). \tag{28}$$

The form given by (24) may be called an "advective form" and the form given by (25) may be called a "flux form," and both are identical in the non-divergent case. The flux form, given by (25), shows that the flux of a quantity from grid point (i, j) to $(i + i', j + j')$ is expressed as the product of the corresponding mass flux and the arithmetic mean of the quantities at the two grid points. The finite difference analogue for the vertical flux of potential temperature, obtained by Lorenz [8], has this form.

Multiplying (22) by $2\zeta_{i,j}$, we get

$$2\zeta_{i,j}\mathbb{J}_{i,j}(\zeta, \psi) = \sum_{i',j'}{}^{*} [2a_{i,j;\,i+i',j+j'}\, \zeta_{i,j}\, \zeta_{i+i',j+j'} - 2a_{i-i',j-j';\,i,j}\, \zeta_{i-i',j-j'}\, \zeta_{i,j}]. \tag{29}$$

It is seen that the right-hand side again consists of the differences of fluxes of the square vorticity in which geometrical means appear, in contrast to the arithmetic means in (25). We see, therefore, that if (20) and (20′) hold, both $\mathbb{J}_{i,j}(\zeta, \psi)$ and $2\zeta_{i,j}\,\mathbb{J}_{i,j}(\zeta, \psi)$ can be properly written in flux forms.

In the usual finite difference scheme for the Jacobian, given by (4), we have

$$a_{i,j;\,i+1,j} = \frac{1}{4d^2}\,(\psi_{i,j+1} - \psi_{i,j-1}), \tag{30.1}$$

$$a_{i,j;\,i-1,j} = -\frac{1}{4d^2}\,(\psi_{i,j+1} - \psi_{i,j-1}), \tag{30.2}$$

$$a_{i,j;\,i,j+1} = -\frac{1}{4d^2}\,(\psi_{i+1,j} - \psi_{i-1,j}), \tag{30.3}$$

$$a_{i,j;\,i,j-1} = \frac{1}{4d^2}\,(\psi_{i+1,j} - \psi_{i-1,j}), \tag{30.4}$$

for arbitrary i and j. Replacing i by $i + 1$ in (30.2), and replacing j by $j + 1$ in (30.4), we get

$$a_{i+1,j;\,i,j} = -\frac{1}{4d^2}\,(\psi_{i+1,j+1} - \psi_{i+1,j-1}), \tag{30.2$'$}$$

and

$$a_{i,j+1;\,i,j} = \frac{1}{4d^2}\,(\psi_{i+1,j+1} - \psi_{i-1,j+1}). \tag{30.4$'$}$$

Comparing (30.1) with (30.2′), and (30.3) with (30.4′), we see that equation (20) is not satisfied by the finite difference scheme given by (4). The net false production of square vorticity, due to the interaction between the grid points (i, j) and $(i + 1, j)$, and that between the grid points (i, j) and $(i, j + 1)$, in this scheme, are

$$2(a_{i,j;\,i+1,j} + a_{i+1,j;\,i,j})\, \zeta_{i,j}\, \zeta_{i+1,j} = -\frac{1}{2d^2}\,[(\psi_{i+1,j+1} - \psi_{i+1,j-1}) - (\psi_{i,j+1} - \psi_{i,j-1})]\, \zeta_{i,j}\, \zeta_{i+1,j},$$

and

$$2(a_{i,j;\,i,j+1} + a_{i,j+1;\,i,j})\,\zeta_{i,j}\,\zeta_{i,j+1}$$
$$= \frac{1}{2d^2}\,[(\psi_{i+1,j+1} - \psi_{i-1,j+1}) - (\psi_{i+1,j} - \psi_{i-1,j})]\,\zeta_{i,j}\,\zeta_{i,j+1}\,.$$

These can be rewritten as

$$\tfrac{1}{2}\,(D_{i+1/2,j+1/2} + D_{i+1/2,j-1/2})\,\zeta_{i,j}\,\zeta_{i+1,j}\,,$$

and

$$-\tfrac{1}{2}\,(D_{i+1/2,j+1/2} + D_{i-1/2,j+1/2})\,\zeta_{i,j}\,\zeta_{i,j+1}\,,$$

where

$$D_{i+1/2,j+1/2} \equiv -\frac{1}{d^2}\,(\psi_{i+1,j+1} + \psi_{i,j} - \psi_{i,j+1} - \psi_{i+1,j})$$

is a finite difference analogue of $-\partial^2\psi/\partial x\partial y$, which is a component of the deformation tensor. Furthermore, the false production of square vorticity, for which $D_{i+1/2,j+1/2}$ is responsible, is expressed as

$$\tfrac{1}{2}\,D_{i+1/2,j+1/2}\,(\zeta_{i,j}\,\zeta_{i+1,j} + \zeta_{i,j+1}\,\zeta_{i+1,j+1} - \zeta_{i,j}\,\zeta_{i,j+1} - \zeta_{i+1,j}\,\zeta_{i+1,j+1}) = -\tfrac{1}{4}\,D_{i+1/2,j+1/2}\,[(\zeta_{i+1,j} - \zeta_{i,j})^2 + (\zeta_{i+1,j+1} - \zeta_{i,j+1})^2 - (\zeta_{i,j+1} - \zeta_{i,j})^2 - (\zeta_{i+1,j+1} - \zeta_{i+1,j})^2],$$

which is a finite difference analogue of the quantity

$$\frac{1}{2}\,\frac{\partial^2\psi}{\partial x\partial y}\,d^2\left[\left(\frac{\partial\zeta}{\partial x}\right)^2 - \left(\frac{\partial\zeta}{\partial y}\right)^2\right] \tag{31}$$

computed for the point $(i + \frac{1}{2}, j + \frac{1}{2})$. If higher order terms in the grid size, d, are neglected, then the expression (31) gives a measure of the false production of square vorticity. Whether the total square vorticity for the whole domain increases or decreases depends on whether the correlation between $\partial^2\psi/\partial x\partial y$ and $(\partial\zeta/\partial x)^2 - (\partial\zeta/\partial y)^2$ is positive or negative. However, solutions of the vorticity equation seem to prefer a positive correlation. For example, where $\partial^2\psi/\partial x\partial y = \partial v/\partial y = -\partial u/\partial x > 0$, an eddy tends to shrink in the x-direction and stretch in the y-direction, causing the magnitude of the vorticity gradient in the x-direction to be larger than the magnitude in the y-direction, and this gives a positive correlation.

The general form of the finite difference analogue of the Jacobian at grid point (i, j), given by (15), may also be rewritten as

$$\mathbb{J}_{i,j}(\zeta, \psi) = \sum_{i'',j''} b_{i,j;\,i+i'',j+j''}\,\psi_{i+i'',j+j''}, \tag{32}$$

where

$$b_{i,j;\,i+i'',j+j''} \equiv \sum_{i',j'} c_{i,j;\,i',j';\,i'',j''}\,\zeta_{i+i',j+j'}\,. \tag{33}$$

Corresponding to (18), we obtain

$$\sum_{i'',j''} b_{i,j;\,i+i'',j+j''} = 0. \tag{34}$$

Furthermore, corresponding to (20), the integral constraint, $\overline{\psi J(\zeta, \psi)} = 0$, which results in the conservation of the kinetic energy in a closed domain, is simulated by requiring that

$$b_{i+i'',j+j'';\,i,j} = -b_{i,j;\,i+i'',j+j''} \tag{35}$$

in the finite difference scheme. The usual difference scheme, given by (4), does not satisfy this requirement and therefore it does not conserve kinetic energy.

For simplicity, let us now consider the following four basic second order finite difference analogues for a square grid:

$$\mathbb{J}^{++}_{i,j}(\zeta, \psi) = \frac{1}{4d^2}\,[(\zeta_{i+1,j} - \zeta_{i-1,j})\,(\psi_{i,j+1} - \psi_{i,j-1}) - (\zeta_{i,j+1} - \zeta_{i,j-1})\,(\psi_{i+1,j} - \psi_{i-1,j})], \tag{36}$$

$$\mathbb{J}^{+\times}_{i,j}(\zeta, \psi) = \frac{1}{4d^2}\,[\zeta_{i+1,j}(\psi_{i+1,j+1} - \psi_{i+1,j-1}) - \zeta_{i-1,j}(\psi_{i-1,j+1} - \psi_{i-1,j-1}) - \zeta_{i,j+1}(\psi_{i+1,j+1} - \psi_{i-1,j+1}) + \zeta_{i,j-1}(\psi_{i+1,j-1} - \psi_{i-1,j-1})], \tag{37}$$

$$\mathbb{J}^{\times+}_{i,j}(\zeta, \psi) = \frac{1}{4d^2}\,[\zeta_{i+1,j+1}(\psi_{i,j+1} - \psi_{i+1,j}) - \zeta_{i-1,j-1}(\psi_{i-1,j} - \psi_{i,j-1}) - \zeta_{i-1,j+1}(\psi_{i,j+1} - \psi_{i-1,j}) + \zeta_{i+1,j-1}\,(\psi_{i+1,j} - \psi_{i,j-1})], \tag{38}$$

$$\mathbb{J}^{\times\times}_{i,j}(\zeta, \psi) = \frac{1}{8d^2}\,[(\zeta_{i+1,j+1} - \zeta_{i-1,j-1})\,(\psi_{i-1,j+1} - \psi_{i+1,j-1}) - (\zeta_{i-1,j+1} - \zeta_{i+1,j-1})\,(\psi_{i+1,j+1} - \psi_{i-1,j-1})]. \tag{39}$$

All four of these finite difference Jacobians maintain the integral constraint given by (5) and all have the same order of accuracy, as we shall see in Section III. More general finite difference analogues for the Jacobian may be obtained by linear combinations of these four basic Jacobians. Thus we put

$$\begin{aligned}\mathbb{J}_{i,j}(\zeta,\psi) &= \alpha\,\mathbb{J}^{++}_{i,j}(\zeta,\psi) + \beta\,\mathbb{J}^{+\times}_{i,j}(\zeta,\psi) \\ &\quad + \gamma\,\mathbb{J}^{\times+}_{i,j}(\zeta,\psi) + \delta\,\mathbb{J}^{\times\times}_{i,j}(\zeta,\psi),\end{aligned} \tag{40}$$

where $\alpha + \beta + \gamma + \delta = 1$.

For this Jacobian, we have

$$a_{i,j;\,i+1,j} = \frac{1}{4d^2}\left[\alpha(\psi_{i,j+1} - \psi_{i,j-1}) + \beta(\psi_{i+1,j+1} - \psi_{i+1,j-1})\right], \tag{41.1}$$

$$a_{i,j;\,i-1,j} = \frac{1}{4d^2}\left[-\alpha(\psi_{i,j+1} - \psi_{i,j-1}) - \beta(\psi_{i-1,j+1} - \psi_{i-1,j-1})\right], \tag{41.2}$$

$$a_{i,j;\,i,j+1} = \frac{1}{4d^2}\left[-\alpha(\psi_{i+1,j} - \psi_{i-1,j}) - \beta(\psi_{i+1,j+1} - \psi_{i-1,j+1})\right], \tag{41.3}$$

$$a_{i,j;\,i,j-1} = \frac{1}{4d^2}\left[\alpha(\psi_{i+1,j} - \psi_{i-1,j}) + \beta(\psi_{i+1,j-1} - \psi_{i-1,j-1})\right], \tag{41.4}$$

$$a_{i,j;\,i+1,j+1} = \frac{1}{4d^2}\left[\gamma(\psi_{i,j+1} - \psi_{i+1,j}) + \frac{\delta}{2}(\psi_{i-1,j+1} - \psi_{i+1,j-1})\right], \tag{41.5}$$

$$a_{i,j;\,i-1,j-1} = \frac{1}{4d^2}\left[-\gamma(\psi_{i-1,j} - \psi_{i,j-1}) - \frac{\delta}{2}(\psi_{i-1,j+1} - \psi_{i+1,j-1})\right], \tag{41.6}$$

$$a_{i,j;\,i-1,j+1} = \frac{1}{4d^2}\left[-\gamma(\psi_{i,j+1} - \psi_{i-1,j}) - \frac{\delta}{2}(\psi_{i+1,j+1} - \psi_{i-1,j-1})\right], \tag{41.7}$$

$$a_{i,j;\,i+1,j-1} = \frac{1}{4d^2}\left[\gamma(\psi_{i+1,j} - \psi_{i,j-1}) + \frac{\delta}{2}(\psi_{i+1,j+1} - \psi_{i-1,j-1})\right]; \tag{41.8}$$

From (41.2), (41.4), (41.6), and (41.8), respectively, we get

$$a_{i+1,j;\,i,j} = \frac{1}{4d^2}\left[-\alpha(\psi_{i+1,j+1} - \psi_{i+1,j-1}) - \beta(\psi_{i,j+1} - \psi_{i,j-1})\right], \tag{41.2$'$}$$

$$a_{i,j+1;\,i,j} = \frac{1}{4d^2}\left[\alpha(\psi_{i+1,j+1} - \psi_{i-1,j+1}) + \beta(\psi_{i+1,j} - \psi_{i-1,j})\right], \tag{41.4$'$}$$

$$a_{i+1,j+1;\,i,j} = \frac{1}{4d^2}\left[-\gamma(\psi_{i,j+1} - \psi_{i+1,j}) - \frac{\delta}{2}(\psi_{i,j+2} - \psi_{i+2,j})\right], \tag{41.6$'$}$$

$$a_{i-1,j+1;\,i,j} = \frac{1}{4d^2}\left[\gamma(\psi_{i,j+1} - \psi_{i-1,j}) + \frac{\delta}{2}(\psi_{i,j+2} - \psi_{i-2,j})\right]. \tag{41.8$'$}$$

Comparison of (41.2′) with (41.1), (41.4′) with (41.3), (41.6′) with (41.5), and (41.8′) with (41.7) reveals that

$$\alpha = \beta, \quad \delta = 0 \tag{42}$$

are required in order to satisfy (20). Thus, the scheme $\alpha[\mathbb{J}^{++}_{i,j}(\zeta,\psi) + \mathbb{J}^{+\times}_{i,j}(\zeta,\psi)] + \gamma\mathbb{J}^{\times+}_{i,j}(\zeta,\psi)$, where $2\alpha + \gamma = 1$ is a square vorticity conserving scheme.

In a similar way, it can be shown that

$$\alpha = \gamma, \quad \delta = 0 \tag{43}$$

are required in order to satisfy (35). Thus, the scheme $\alpha[\mathbb{J}^{++}_{i,j}(\zeta,\psi) + \mathbb{J}^{\times+}_{i,j}(\zeta,\psi)] + \beta\mathbb{J}^{+\times}_{i,j}(\zeta,\psi)$, where $2\alpha + \beta = 1$ is an energy conserving scheme.

The scheme which satisfies both the conservation of square vorticity and the conservation of energy is given by

$$\alpha = \beta = \gamma = \tfrac{1}{3}, \quad \delta = 0. \tag{44}$$

By the choice of the coefficients α, β, γ, and δ, one can obtain, among others, the forms of the Jacobian in Table I. The table shows which of these typical Jacobians, which are sometimes used, satisfy $J(\zeta,\psi) = -J(\psi,\zeta)$ or conserve the mean square vorticity or the mean kinetic energy. Only the linear combination $[\mathbb{J}^{++}_{i,j}(\zeta,\psi) + \mathbb{J}^{+\times}_{i,j}(\zeta,\psi) + \mathbb{J}^{\times+}_{i,j}(\zeta,\psi)]/3$ will satisfy $J(\zeta,\psi) = -J(\psi,\zeta)$ and also conserve both of the quadratic quantities.

This last scheme can be written as

$$\begin{aligned}\mathbb{J}_{i,j}(\zeta,\psi) = -\frac{1}{12d^2}[&(\psi_{i,j-1} + \psi_{i+1,j-1} - \psi_{i,j+1} - \psi_{i+1,j+1}) \\ &(\zeta_{i+1,j} - \zeta_{i,j}) + (\psi_{i-1,j-1} + \psi_{i,j-1} - \psi_{i-1,j+1} - \psi_{i,j+1}) \\ &(\zeta_{i,j} - \zeta_{i-1,j}) + (\psi_{i+1,j} + \psi_{i+1,j+1} - \psi_{i-1,j} - \psi_{i-1,j+1}) \\ &(\zeta_{i,j+1} - \zeta_{i,j}) + (\psi_{i+1,j-1} + \psi_{i+1,j} - \psi_{i-1,j-1} - \psi_{i-1,j}) \\ &(\zeta_{i,j} - \zeta_{i,j-1}) + (\psi_{i+1,j} - \psi_{i,j+1})(\zeta_{i+1,j+1} - \zeta_{i,j})\end{aligned}$$

TABLE I

Properties of Typical Jacobians

$J(\zeta, \psi) \Rightarrow$	$\mathbb{J}^{++}$	$\mathbb{J}^{+\times}$	$\mathbb{J}^{\times+}$	$\frac{\mathbb{J}^{++} + \mathbb{J}^{+\times}}{2}$	$\frac{\mathbb{J}^{+\times} + \mathbb{J}^{\times+}}{2}$	$\frac{\mathbb{J}^{\times+} + \mathbb{J}^{++}}{2}$	$\frac{\mathbb{J}^{++} + \mathbb{J}^{+\times} + \mathbb{J}^{\times+}}{3}$
$J(\zeta, \psi) = -J(\psi, \zeta)$	√[a]				√		√
Square vorticity conserved			√	√			√
Kinetic energy conserved		√				√	√

[a] A check mark indicates that the property in the left-hand column is maintained.

$$+ (\psi_{i,j-1} - \psi_{i-1,j})(\zeta_{i,j} - \zeta_{i-1,j-1}) + (\psi_{i,j+1} - \psi_{i-1,j})$$
$$(\zeta_{i-1,j+1} - \zeta_{i,j}) + (\psi_{i+1,j} - \psi_{i,j-1})(\zeta_{i,j} - \zeta_{i+1,j-1})], \tag{45}$$

or

$$\begin{aligned}\mathbb{J}_{i,j}(\zeta, \psi) = -\frac{1}{12d^2}&[(\psi_{i,j-1} + \psi_{i+1,j-1} - \psi_{i,j+1} - \psi_{i+1,j+1})\\ &(\zeta_{i+1,j} + \zeta_{i,j}) - (\psi_{i-1,j-1} + \psi_{i,j-1} - \psi_{i-1,j+1} - \psi_{i,j+1})\\ &(\zeta_{i,j} + \zeta_{i-1,j}) + (\psi_{i+1,j} + \psi_{i+1,j+1} - \psi_{i-1,j} - \psi_{i-1,j+1})\\ &(\zeta_{i,j+1} + \zeta_{i,j}) - (\psi_{i+1,j-1} + \psi_{i+1,j} - \psi_{i-1,j-1} - \psi_{i-1,j})\\ &(\zeta_{i,j} + \zeta_{i,j-1}) + (\psi_{i+1,j} - \psi_{i,j+1})(\zeta_{i+1,j+1} + \zeta_{i,j})\\ &- (\psi_{i,j-1} - \psi_{i-1,j})(\zeta_{i,j} + \zeta_{i-1,j-1}) + (\psi_{i,j+1} - \psi_{i-1,j})\\ &(\zeta_{i-1,j+1} + \zeta_{i,j}) - (\psi_{i+1,j} - \psi_{i,j-1})(\zeta_{i,j} + \zeta_{i+1,j-1})].\end{aligned} \tag{46}$$

Equations (45) and (46) correspond to the advective form and the flux form of the Jacobian, given by (24) and (25), respectively. The property of the Jacobian $\mathbb{J}_{i,j}(\zeta, \psi) = -\mathbb{J}_{i,j}(\psi, \zeta)$, which requires that $\beta = \gamma$, is automatically satisfied.

III. ACCURACY OF THE DIFFERENCE SCHEME

Since the finite difference scheme for the Jacobian, given by (45) or (46), is a linear combination of the basic second order finite difference schemes (36), (37), and (38), we can expect that this scheme has an accuracy of the same order as that of the basic schemes. Expanding ζ and ψ into Taylor series around the point (i, j), we have

$$\begin{aligned}\mathbb{J}^{++}(\zeta, \psi) &= J(\zeta, \psi)\\ &+ \frac{d^2}{6}\left[\frac{\partial\zeta}{\partial x}\frac{\partial^3\psi}{\partial y^3} - \frac{\partial\zeta}{\partial y}\frac{\partial^3\psi}{\partial x^3} + \frac{\partial^3\zeta}{\partial x^3}\frac{\partial\psi}{\partial y} - \frac{\partial^3\zeta}{\partial y^3}\frac{\partial\psi}{\partial x}\right]\\ &+ O(d^4),\end{aligned} \tag{47}$$

$$\begin{aligned}\mathbb{J}^{+\times}(\zeta, \psi) &= J(\zeta, \psi)\\ &+ \frac{d^2}{6}\left[\frac{\partial\zeta}{\partial x}\frac{\partial^3\psi}{\partial y^3} - \frac{\partial\zeta}{\partial y}\frac{\partial^3\psi}{\partial x^3} + \frac{\partial^3\zeta}{\partial x^3}\frac{\partial\psi}{\partial y} - \frac{\partial^3\zeta}{\partial y^3}\frac{\partial\psi}{\partial x}\right.\\ &+ 3\left(\frac{\partial\zeta}{\partial x}\frac{\partial^3\psi}{\partial x^2\partial y} - \frac{\partial\zeta}{\partial y}\frac{\partial^3\psi}{\partial x\partial y^2}\right)\\ &\left.+ 3\left(\frac{\partial^2\zeta}{\partial x^2} - \frac{\partial^2\zeta}{\partial y^2}\right)\frac{\partial^2\psi}{\partial x\,\partial y}\right] + O(d^4),\end{aligned} \tag{48}$$

$$\begin{aligned}\mathbb{J}^{\times+}(\zeta, \psi) &= J(\zeta, \psi)\\ &+ \frac{d^2}{6}\left[\frac{\partial\zeta}{\partial x}\frac{\partial^3\psi}{\partial y^3} - \frac{\partial\zeta}{\partial y}\frac{\partial^3\psi}{\partial x^3} + \frac{\partial^3\zeta}{\partial x^3}\frac{\partial\psi}{\partial y} - \frac{\partial^3\zeta}{\partial y^3}\frac{\partial\psi}{\partial x}\right.\\ &- 3\left(\frac{\partial\psi}{\partial x}\frac{\partial^3\zeta}{\partial x^2\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial^3\zeta}{\partial x\partial y^2}\right)\\ &\left.- 3\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right)\frac{\partial^2\zeta}{\partial x\,\partial y}\right] + O(d^4),\end{aligned} \tag{49}$$

where the subscripts i, j are omitted. Since our Jacobian, denoted by $\mathbb{J}_1(\zeta, \psi)$ in this section, is given by $[\mathbb{J}^{++}(\zeta, \psi) + \mathbb{J}^{+\times}(\zeta, \psi) + \mathbb{J}^{\times+}(\zeta, \psi)]/3$, we have

$$\begin{aligned}\mathbb{J}_1(\zeta, \psi) &= J(\zeta, \psi)\\ &+ \frac{d^2}{6}\left[\frac{\partial\zeta}{\partial x}\frac{\partial^3\psi}{\partial y^3} - \frac{\partial\zeta}{\partial y}\frac{\partial^3\psi}{\partial x^3} + \frac{\partial^3\zeta}{\partial x^3}\frac{\partial\psi}{\partial y} - \frac{\partial^3\zeta}{\partial y^3}\frac{\partial\psi}{\partial x}\right.\\ &+ \left(\frac{\partial\zeta}{\partial x}\frac{\partial^3\psi}{\partial x^2\partial y} - \frac{\partial\zeta}{\partial y}\frac{\partial^3\psi}{\partial x\partial y^2}\right) + \left(\frac{\partial^2\zeta}{\partial x^2} - \frac{\partial^2\zeta}{\partial y^2}\right)\frac{\partial^2\psi}{\partial x\,\partial y}\end{aligned} \tag{50}$$

$$- \left(\frac{\partial \psi}{\partial x} \frac{\partial^3 \zeta}{\partial x^2 \partial y} - \frac{\partial \psi}{\partial y} \frac{\partial^3 \zeta}{\partial x \partial y^2} \right) - \left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} \right) \frac{\partial^2 \zeta}{\partial x\, \partial y} \Bigg]$$

$$+ O(d^4).$$

To examine a phase error, we consider a simple stream function

$$\psi = -UY + A \sin kX, \tag{51}$$

where (X, Y) are rectangular coordinates obtained by the rotation of the coordinate axis through the angle θ. That is,

$$\left. \begin{aligned} X &= x \cos\theta + y \sin\theta \\ Y &= -x \sin\theta + y \cos\theta \end{aligned} \right\}. \tag{52}$$

The vorticity is given by

$$\zeta = -Ak^2 \sin kX. \tag{53}$$

In a finite difference calculation, the vorticity is also expressed in a finite difference form; but, here, the exact form (53) is used in order to estimate the error resulting only from the finite difference scheme for the Jacobian.

The error in the usual scheme $\mathbb{J}^{++}(\zeta, \psi)$, given by (47), is

$$U \frac{\partial \zeta}{\partial X} \frac{(kd)^2}{6} (\cos^4\theta + \sin^4\theta) + O(d^4). \tag{54}$$

By contrast, the error in the scheme $\mathbb{J}_1(\zeta, \psi)$, given by (50), is

$$U \frac{\partial \zeta}{\partial X} \frac{(kd)^2}{6} (\cos^2\theta + \sin^2\theta)^2 + O(d^4). \tag{55}$$

In the range $0 \leqq \theta \leqq \pi/2$, the factor $(\cos^4\theta + \sin^4\theta)$ in (54) has the maximum value 1 at $\theta = 0$ and $\theta = \pi/2$ and has the minimum value 1/2 at $\theta = \pi/4$. On the other hand, the factor $(\cos^2\theta + \sin^2\theta)^2$ in (55) is always 1, which means the orientation error is removed in this case. Although the error in this scheme is larger around $\theta = \pi/4$, it does not exceed the maximum value of the error in the ordinary scheme.

There are many other schemes, in addition to $\mathbb{J}_1(\zeta, \psi)$, which conserve the square of vorticity and the energy. For example, if we use the additional grid points $(i+2, j)$, $(i-2, j)$, $(i, j+2)$, and $(i, j-2)$, and other scheme, $\mathbb{J}_2(\zeta, \psi)$, defined by

$$\mathbb{J}_2(\zeta, \psi) = \tfrac{1}{3} [\mathbb{J}^{\times\times}(\zeta, \psi) + \mathbb{J}^{\times+}(\zeta, \psi) + \mathbb{J}^{+\times}(\zeta, \psi)], \tag{56}$$

also conserves the square of vorticity and the energy. Here $\mathbb{J}^{\times\times}(\zeta, \psi)$ is defined by (39) and

$$\begin{aligned} \mathbb{J}^{\times+}_{i,j}(\zeta, \psi) = \frac{1}{8d^2} [&\zeta_{i+1,j+1}(\psi_{i,j+2} - \psi_{i+2,j}) \\ &- \zeta_{i-1,j-1}(\psi_{i-2,j} - \psi_{i,j-2}) \\ &- \zeta_{i-1,j+1}(\psi_{i,j+2} - \psi_{i-2,j}) \\ &+ \zeta_{i+1,j-1}(\psi_{i+2,j} - \psi_{i,j-2})], \end{aligned} \tag{57}$$

$$\begin{aligned} \mathbb{J}^{+\times}_{i,j}(\zeta, \psi) = \frac{1}{8d^2} [&\zeta_{i+2,j}(\psi_{i+1,j+1} - \psi_{i+1,j-1}) \\ &- \zeta_{i-2,j}(\psi_{i-1,j+1} + \psi_{i-1,j-1}) \\ &- \zeta_{i,j+2}(\psi_{i+1,j+1} - \psi_{i-1,j+1}) \\ &+ \zeta_{i,j-2}(\psi_{i+1,j-1} - \psi_{i-1,j-1})]. \end{aligned} \tag{58}$$

The accuracy of $\mathbb{J}_2(\zeta, \psi)$ is given by

$$\begin{aligned} \mathbb{J}_2(\zeta, \psi) = J(\zeta, \psi) \\ &+ \frac{d^2}{3} \Bigg[\frac{\partial \zeta}{\partial x} \frac{\partial^3 \psi}{\partial y^3} - \frac{\partial \zeta}{\partial y} \frac{\partial^3 \psi}{\partial x^3} + \frac{\partial^3 \zeta}{\partial x^3} \frac{\partial \psi}{\partial y} - \frac{\partial^3 \zeta}{\partial y^3} \frac{\partial \psi}{\partial x} \\ &+ \left(\frac{\partial \zeta}{\partial x} \frac{\partial^3 \psi}{\partial x^2 \partial y} - \frac{\partial \zeta}{\partial y} \frac{\partial^3 \psi}{\partial x \partial y^2} \right) + \left(\frac{\partial^2 \zeta}{\partial x^2} - \frac{\partial^2 \zeta}{\partial y^2} \right) \frac{\partial^2 \psi}{\partial x\, \partial y} \\ &- \left(\frac{\partial \psi}{\partial x} \frac{\partial^3 \zeta}{\partial x^2 \partial y} - \frac{\partial \psi}{\partial y} \frac{\partial^3 \zeta}{\partial x \partial y^2} \right) - \left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} \right) \\ &\frac{\partial^2 \zeta}{\partial x\, \partial y} \Bigg] + O(d^4). \end{aligned} \tag{59}$$

From (50) and (59), we see that $2\mathbb{J}_1(\zeta, \psi) - \mathbb{J}_2(\zeta, \psi)$ is a fourth order approximation of the Jacobian; that is,

$$2\mathbb{J}_1(\zeta, \psi) - \mathbb{J}_2(\zeta, \psi) = J(\zeta, \psi) + O(d^4). \tag{60}$$

IV. CURVILINEAR GRIDS AND BOUNDARY CONDITIONS

Consider an orthogonal curvilinear coordinate system, (ξ, η). Define m and n as

$$m = \frac{\delta\xi}{(\delta s)_\xi}, \quad n = \frac{\delta\eta}{(\delta s)_\eta}, \tag{61}$$

where $(\delta s)_\xi$ is the increment of distance for a change of $\delta\xi$ in ξ, and $(\delta s)_\eta$ is the increment of distance for a change of $\delta\eta$ in η. The wind components in the ξ-direction and η-direction are

$$u = \frac{1}{m}\frac{d\xi}{dt}, \quad v = \frac{1}{n}\frac{d\eta}{dt}, \tag{62}$$

respectively. Divergence and vorticity are

$$\nabla \cdot \mathbf{v} = mn\left[\frac{\partial}{\partial \xi}\left(\frac{u}{n}\right) + \frac{\partial}{\partial \eta}\left(\frac{v}{m}\right)\right], \tag{63}$$

and

$$\mathbf{k} \cdot \nabla \times \mathbf{v} = mn\left[\frac{\partial}{\partial \xi}\left(\frac{v}{n}\right) - \frac{\partial}{\partial \eta}\left(\frac{u}{m}\right)\right]. \tag{64}$$

The vorticity equation (1) becomes

$$\frac{\partial \zeta}{\partial t} = -mn\left[\frac{u}{n}\frac{\partial \zeta}{\partial \xi} + \frac{v}{m}\frac{\partial \zeta}{\partial \eta}\right]. \tag{65}$$

Since $\nabla \cdot \mathbf{v} = 0$ in two-dimensional incompressible flow, we define a stream function by

$$\frac{u}{n} = -\frac{\partial \psi}{\partial \eta}, \quad \frac{v}{m} = \frac{\partial \psi}{\partial \xi}. \tag{66}$$

The vorticity equation (65) can be rewritten as

$$\frac{\partial}{\partial t}\left(\frac{\zeta}{mn}\right) = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial \xi}\left(\frac{v}{n}\right) - \frac{\partial}{\partial \eta}\left(\frac{u}{m}\right)\right] = J(\zeta, \psi), \tag{67}$$

where

$$J(p, q) = \frac{\partial p}{\partial \xi}\frac{\partial q}{\partial \eta} - \frac{\partial p}{\partial \eta}\frac{\partial q}{\partial \xi}. \tag{68}$$

The integral constraints (5), (6), and (7) hold for this Jacobian, if the bar is redefined as the average over the domain in the (ξ, η) plane. We see from (67) that the conservations of mean vorticity, $\overline{\zeta/mn}$, mean square vorticity, $\overline{\zeta^2/mn}$, and mean kinetic energy, $\overline{(u^2 + v^2)/2mn}$, in a closed domain, along the boundary of which ψ is constant, again result from these integral constraints on the Jacobian. Therefore, for a square grid, in the ξ, η plane, which has the grid interval $\Delta\xi = \Delta\eta = d$, the same difference scheme that was derived above can be applied to the right-hand side of (67).

In order to maintain the integral constraints in a bounded domain, the boundary must be treated properly. For simplicity, let us consider a domain, bounded by the coordinate lines $\eta = \eta_0$ and $\eta = \eta_J = \eta_0 + Jd$. A cyclic change is assumed in the ξ-direction. If the domain is *closed*, in the sense that $(v/m)_{\eta=\eta_0}$ and $(v/m)_{\eta=\eta_J}$ are zero, ψ is constant along each of the boundaries. We define indices i, j as $\xi = id$ and $\eta = \eta_0 + jd$. The boundaries are $j = 0$ and $j = J$.

If the finite difference scheme derived in Section II is used for the term on the right in (67) at the inner grid points, then

$$\frac{\partial}{\partial t}\left(\frac{\zeta}{mn}\right)_{i,j} = \mathbb{J}_{i,j}(\zeta, \psi) \tag{69}$$

for $j = 1, 2, \ldots, J - 1$, where $\mathbb{J}_{i,j}(\zeta, \psi)$ is given by (46). The area represented by the grid point (i, j) is $(d^2/mn)i, j$.

Consider, first, the case where m is finite at the boundary. Let the areas represented by the grid points $(i, 0)$ and (i, J) be $(d^2/2mn)_{i,0}$ and $(d^2/2mn)_{i,J}$, respectively.

For example, let us consider the boundary $j = 0$. Since the scheme given by (69) is used at grid point $(i, 1)$, and the general form of the vorticity and the square vorticity conserving scheme can be written as (25), we can write

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\zeta}{mn}\right)_{i,0} = {} & a_{i,0;\,i+1,0}(\zeta_{i,0} + \zeta_{i+1,0}) - a_{i-1,0;\,i,0}(\zeta_{i-1,0} + \zeta_{i,0}) \\ & - \frac{1}{12d^2}[(\psi_{i+1,0} + \psi_{i+1,1} - \psi_{i-1,0} - \psi_{i-1,1}) \\ & (\zeta_{i,0} + \zeta_{i,1}) + (\psi_{i+1,0} - \psi_{i,1})(\zeta_{i,0} + \zeta_{i+1,1}) \\ & + (\psi_{i,1} - \psi_{i-1,0})(\zeta_{i-1,1} + \zeta_{i,0})].\end{aligned} \tag{70}$$

Corresponding to (18) we have

$$\begin{aligned}&(a_{i,0;\,i+1,0} - a_{i-1,0;\,i,0}) \\ &\quad - \frac{1}{12d^2}[2\psi_{i+1,0} - 2\psi_{i-1,0} + \psi_{i+1,1} - \psi_{i-1,1}] = 0.\end{aligned} \tag{71}$$

As $\psi_{i,0}$ does not depend on i, (71) can be rewritten as

$$\begin{aligned}a_{i,0;\,i+1,0} &- \frac{1}{12d^2}(\psi_{i,1} + \psi_{i+1,1} - \psi_{i,0} - \psi_{i+1,0}) \\ &= a_{i-1,0;\,i,0} - \frac{1}{12d^2}(\psi_{i-1,1} + \psi_{i,1} - \psi_{i-1,0} - \psi_{i,0}).\end{aligned} \tag{72}$$

Since the right-hand side of (72) is obtained by replacing i by $i - 1$ in the left-hand side of (72), the quantity $a_{i,0;\,i+1,0} - (\psi_{i,1} + \psi_{i+1,1} - \psi_{i,0} - \psi_{i+1,0})/12d^2$ is a constant which does not depend on i. Because the right-hand side of (70) must approach $-(u/2n)(\partial\zeta/\partial\xi)$, it can be shown that the constant must be zero.

Equation (70) becomes

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\zeta}{mn}\right)_{i,0} \\
&\quad = -\frac{1}{12d^2}[(\psi_{i,0}+\psi_{i+1,0}-\psi_{i,1}-\psi_{i+1,1})(\zeta_{i,0}+\zeta_{i+1,0}) \\
&\qquad -(\psi_{i-1,0}+\psi_{i,0}-\psi_{i-1,1}-\psi_{i,1})(\zeta_{i-1,0}+\zeta_{i,0}) \\
&\qquad +(\psi_{i+1,0}+\psi_{i+1,1}-\psi_{i-1,0}-\psi_{i-1,1})(\zeta_{i,0}+\zeta_{i,1}) \\
&\qquad +(\psi_{i+1,0}-\psi_{i,1})(\zeta_{i,0}+\zeta_{i+1,1}) \\
&\qquad +(\psi_{i,1}-\psi_{i-1,0})(\zeta_{i-1,1}+\xi_{i,0})],
\end{aligned} \tag{73}$$

where

$$\psi_{i,0} = \Psi_0, \tag{73$'$}$$

for all i.

Similarly, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\zeta}{mn}\right)_{i,J} \\
&\quad = -\frac{1}{12d^2}[(\psi_{i,J-1}+\psi_{i+1,J-1}-\psi_{i,J}-\psi_{i+1,J})(\zeta_{i,J}+\zeta_{i+1,J}) \\
&\qquad -(\psi_{i-1,J-1}+\psi_{i,J-1}-\psi_{i-1,J}-\psi_{i,J})(\zeta_{i-1,J}+\zeta_{i,J}) \\
&\qquad -(\psi_{i+1,J-1}+\psi_{i+1,J}-\psi_{i-1,J-1}-\psi_{i-1,J})(\zeta_{i,J-1}+\zeta_{i,J}) \\
&\qquad -(\psi_{i,J-1}-\psi_{i-1,J})(\zeta_{i-1,J-1}+\zeta_{i,J}) \\
&\qquad -(\psi_{i+1,J}-\psi_{i,J-1})(\zeta_{i,J}+\zeta_{i+1,J-1})],
\end{aligned} \tag{74}$$

where

$$\psi_{i,J} = \Psi_J, \tag{74$'$}$$

for all i. Ψ_0 and Ψ_J are functions of time only.

The conservation of vorticity takes the form

$$\sum_i\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\zeta}{mn}\right)_{i,0}+\sum_{j=1}^{J-1}\frac{\partial}{\partial t}\left(\frac{\zeta}{mn}\right)_{i,j}+\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\zeta}{mn}\right)_{i,J}\right]=0. \tag{75}$$

Consider, now, the finite difference analogues of the vorticity, ζ, and the wind components, u and v, given by

$$\begin{aligned}
\left(\frac{\zeta}{mn}\right)_{i,j} = \frac{1}{d}\Bigg[&\left(\frac{v}{n}\right)_{i+1/2,j}-\left(\frac{v}{n}\right)_{i-1/2,j} \\
&+\left(\frac{u}{m}\right)_{i,j-1/2}-\left(\frac{u}{m}\right)_{i,j+1/2}\Bigg]
\end{aligned} \tag{76}$$

$$\left(\frac{u}{n}\right)_{i,j+1/2} = \frac{1}{d}(\psi_{i,j}-\psi_{i,j+1}), \tag{77}$$

$$\left(\frac{v}{m}\right)_{i+1/2,j} = \frac{1}{d}(\psi_{i+1,j}-\psi_{i,j}). \tag{78}$$

From (75) and (76), we can put

$$\sum_i\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\zeta}{mn}\right)_{i,0} = -\frac{1}{d}\sum_i\frac{\partial}{\partial t}\left(\frac{u}{m}\right)_{i,1/2}, \tag{79.1}$$

$$\sum_i\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\zeta}{mn}\right)_{i,J} = \frac{1}{d}\sum_i\frac{\partial}{\partial t}\left(\frac{u}{m}\right)_{i,J-1/2}. \tag{79.2}$$

It can be shown that the integral constraint $\overline{\psi J(\zeta,\psi)} = 0$ is also maintained if this mean is replaced by

$$\sum_i\left[\frac{1}{2}\psi_{i,0}\,\mathbb{J}_{i,0}(\zeta,\psi)+\sum_{j=1}^{J-1}\psi_{i,j}\,\mathbb{J}_{i,j}(\zeta,\psi)+\frac{1}{2}\psi_{i,J}\,\mathbb{J}_{i,J}(\zeta,\psi)\right],$$

where $\frac{1}{2}\mathbb{J}_{i,0}(\zeta,\psi)$, $\mathbb{J}_{i,j}(\zeta,\psi)$ and $\frac{1}{2}\mathbb{J}_{i,J}(\zeta,\psi)$ are given by the right-hand sides of (73), (46), and (74), respectively. The finite difference expression for the conservation of the kinetic energy, in this scheme, is written as

$$\begin{aligned}
&-\sum_i\left[\psi_{i,0}\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\zeta}{mn}\right)_{i,0}+\sum_{j=1}^{J-1}\psi_{i,j}\frac{\partial}{\partial t}\left(\frac{\zeta}{mn}\right)_{i,j}+\psi_{i,J}\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\zeta}{mn}\right)_{i,J}\right] \\
&= \frac{1}{d}\Bigg[\psi_{i,0}\sum_i\frac{\partial}{\partial t}\left(\frac{u}{m}\right)_{i,1/2}-\sum_i\sum_{j=1}^{J-1}\psi_{i,j}\Bigg\{\frac{\partial}{\partial t}\left(\frac{v}{n}\right)_{i+1/2,j} \\
&\qquad -\frac{\partial}{\partial t}\left(\frac{v}{n}\right)_{i-1/2,j}+\frac{\partial}{\partial t}\left(\frac{u}{m}\right)_{i,j-1/2}-\frac{\partial}{\partial t}\left(\frac{u}{m}\right)_{i,j+1/2}\Bigg\} \\
&\qquad -\psi_{i,J}\sum_i\frac{\partial}{\partial t}\left(\frac{u}{m}\right)_{i,J-1/2}\Bigg] \\
&= \frac{1}{d}\sum_i\Bigg[\sum_{j=0}^{J-1}(\psi_{i,j}-\psi_{i,j+1})\frac{\partial}{\partial t}\left(\frac{u}{m}\right)_{i,j+1/2} \\
&\qquad +\sum_{j=1}^{J-1}(\psi_{i+1,j}-\psi_{i,j})\frac{\partial}{\partial t}\left(\frac{v}{n}\right)_{i+1/2,j}\Bigg] \\
&= \sum_i\left[\sum_{j=0}^{J-1}\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{u^2}{mn}\right)_{i,j+1/2}+\sum_{j=1}^{J-1}\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{v^2}{mn}\right)_{i+1/2,j}\right]=0.
\end{aligned} \tag{80}$$

Equations (73), (74), (73′), (74′), (79.1), and (79.2) completely define the boundary conditions. In the simple case when m and n are functions of j only, it is convenient to divide the stream function into two parts,

$$\psi_{i,j} = \Psi_j + \psi'_{i,j}, \quad \Psi_j = \overline{\psi_{i,j}}^{\,i}, \tag{81}$$

where $\overline{(\ \ )}^{\,i}$ denotes the mean in i and the prime denotes the deviation. Now, let $m_{i,j} = m_j$ and $n_{i,j} = n_j$. From (79.1) and (79.2), we get

$$\frac{\partial}{\partial t}(\Psi_0 - \Psi_1) = -d^2\left(\frac{m}{n}\right)_{1/2}\frac{\partial}{\partial t}\overline{\left(\frac{1}{2}\frac{\zeta}{mn}\right)_{i,0}}^{\,i}, \tag{82}$$

$$\frac{\partial}{\partial t}(\Psi_{J-1} - \Psi_J) = d^2\left(\frac{m}{n}\right)_{J-1/2}\frac{\partial}{\partial t}\overline{\left(\frac{1}{2}\frac{\zeta}{mn}\right)_{i,J}}^{\,i}, \tag{83}$$

where the time derivatives in the right-hand sides are given by (73) and (74). At the inner points, $0 < j < J$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\left[\left(\frac{n}{m}\right)_{j-1/2}(\Psi_{j-1} - \Psi_j) - \left(\frac{n}{m}\right)_{j+1/2}(\Psi_j - \Psi_{j+1})\right]\\ = d^2\frac{\partial}{\partial t}\overline{\left(\frac{\zeta}{mn}\right)_{i,j}}^{\,i}.\end{aligned} \tag{84}$$

For the deviation part, we have

$$\begin{aligned}\frac{\partial}{\partial t}\Bigg[\left(\frac{n}{m}\right)_{j-1/2}(\psi'_{i,j-1} - \psi'_{i,j}) - \left(\frac{n}{m}\right)_{j+1/2}(\psi'_{i,j} - \psi'_{i,j+1})\\ + \left(\frac{n}{m}\right)_j(\psi'_{i+1,j} + \psi'_{i-1,j} - 2\psi'_{i,j})\Bigg] = d^2\frac{\partial}{\partial t}\left(\frac{\zeta}{mn}\right)'_{i,j}\end{aligned} \tag{85}$$

for $0 < j < J$, and

$$\psi'_{i,0} = 0, \quad \psi'_{i,J} = 0. \tag{86}$$

The time derivatives on the right-hand sides of (84) and (85) are given by (69).

If m becomes infinite at $\eta = \eta_0$, then $\eta = \eta_0$ becomes a singular point, like the pole in a polar coordinate system. In this case, (73) must be replaced by

$$\begin{aligned}\frac{\partial}{\partial t}(\varepsilon\,\zeta_0) = -\frac{1}{12d^2}\sum_{i=1}^{I}\big[(\psi_{i+1,0} + \psi_{i+1,1} - \psi_{i-1,0} - \psi_{i-1,1})\\ (\zeta_0 + \zeta_{i,1}) + (\psi_{i+1,0} - \psi_{i,1})(\zeta_0 + \zeta_{i+1,1})\\ + (\psi_{i,1} - \psi_{i-1,0})(\zeta_{i-1,1} + \zeta_0)\big]\end{aligned} \tag{87}$$

and

$$\psi_{i,0} = \Psi_0 \tag{87$'$}$$

for all i, where ζ_0 and Ψ_0 are functions of time only. I is the number of grid points on the line $\eta = \eta_0 + d$ and εd^2 denotes the area represented by the singular point. In addition, (79.1) must be replaced by

$$\begin{aligned}\frac{\partial}{\partial t}(\varepsilon\,\zeta_0) &= -\frac{1}{d}\sum_{i=1}^{I}\frac{\partial}{\partial t}\left(\frac{u}{m}\right)_{i,1/2}\\ &= -\frac{1}{d^2}\sum_{i=1}^{I}\frac{\partial}{\partial t}\left(\frac{n}{m}\right)_{i,1/2}(\Psi_0 - \psi_{i,1}).\end{aligned} \tag{88}$$

Equations (87), (87′), and (88) completely define the conditions at the singular point.

V. CONCLUSION

It was shown that in two-dimensional incompressible flow some of the integral constraints on quantities of physical importance, such as the conservation of mean kinetic energy, mean square vorticity, (and mean vorticity itself), can be maintained if the finite difference analogue for the advection term is properly designed.

Since the required constraints are on the advection term, which has the form of a Jacobian operator for the flow considered, the finite difference scheme for the Jacobian must have a certain restricted form. Based upon a consistent interaction between grid points, a general form of finite difference Jacobian, which maintains the integral constraints, was derived. Examples were given for the second-order nine-point scheme and the fourth-order thirteen-point scheme. The boundary conditions at a wall, and the conditions at a singular point in a curvilinear grid, which satisfy the integral constraints, were also indicated.

When the quadratic quantities are conserved in a finite difference scheme, nonlinear computational instability cannot occur. This follows from the fact that if the square of a quantity is conserved with time when summed up over all the grid points in a domain, the quantity itself will be bounded, at every individual grid point, throughout the entire period of integration.

Phillips [2] attributed the cause of nonlinear computational instability to "aliasing," or misrepresentation of an unresolvable short wave by a resolvable longer wave in the computed time derivative of the stream function. However, aliasing does not necessarily mean a false production of energy. Whether amplification does or does not occur depends on the phase relation between the misrepresented wave in the time derivative and the wave which is already present.

Aliasing does exist in the finite difference scheme developed in this paper. It may appear as a phase error or as a distortion of the spectral distribution of energy. But the total energy and the average scale of the motion is free from aliasing error in this scheme.

Lilly [9] compared the aliasing error with first derivative errors in a limited component wave system. He used the difference scheme given by (45) and found that the aliasing error was smaller than the first derivative errors. If a uniform or large-scale flow is superposed on such a limited component wave system, as is often done, the first derivative errors become even more serious, while there is no additional aliasing error. The higher order scheme, derived in Section III, will decrease the first derivative errors considerably.

A numerical example, which uses the scheme derived in this paper, will be given in Part II of this paper. Comparisons will be made, there, with the results obtained with the usual space difference scheme, showing not only the stability of the two schemes but also their influence on the spectral distribution of kinetic energy and the frequency distribution of vorticity. The time-differencing problem will also be discussed in Part II.

It is clear that the advective term, in a finite difference scheme, can only transfer properties within the range of scales of motion that are resolved by the grid. The advection term cannot produce any interaction between the grid-scale motions and subgrid-scale motions, which are too small to be resolved by the grid. If we have a certain grid size and wish to simulate by finite difference methods the interaction between the grid-scale motions and some subgrid-scale motions, then we must add to the advective terms of the finite difference scheme some additional terms which represent the physical process of grid-scale subgrid-scale interaction. The additional terms should be determined by physical considerations only, and not by the computational need to absorb any falsely generated energy produced by truncation error.

The finite difference scheme for the Jacobian, derived in this paper for two-dimensional incompressible flow, can be applied to equations for quasi-nondivergent flow, if the advection terms have the form of Jacobians. An example is the quasi-geostrophic system of equations [10], which are the equations of first order approximation for geostrophic motion of type 1 [11], that are valid for the cyclone-scale motions of the atmosphere. In this case, the conservation of the energy (which is now the sum of kinetic energy and available potential energy) and the conservation of the mean square potential vorticity, which are quadratic quantities, are maintained in this finite difference scheme.

The fundamental concept of this paper is that a finite difference scheme expresses the interaction between grid points. This concept can be used to design computing schemes for more general flow. It has already been used to obtain a second order computational scheme for the so-called "primitive equations." This scheme is being used for numerical experiments with the Mintz–Arakawa general circulation model [12].

The higher order scheme, derived in Section III, is being used for quasi-geostrophic numerical weather predictions, on an operational basis in Japan [13].

The scheme which conserves the mean kinetic energy, but not the mean square vorticity, (Eq. (37)), has been used by Lilly [14] for two-dimensional convection studies, and by Bryan [15] for ocean current calculations.

A scheme which conserves the mean square vorticity, but not the kinetic energy, was independently derived by Fromm [16] and has been used by him for computing two-dimensional viscous flow.

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