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An Objective Analysis of Wind Data for Energy Budget Studies

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ABSTRACT

An objective technique for adjusting wind data, such that the total mass divergence in a volume of the atmosphere is zero, is developed. The adjustment is obtained by applying a least-squares smoothing with a Lagrangian multiplier to constrain the total mass divergence to a specified amount. The computational details are derived and the method is applied to several examples. Both for theoretical wind profiles and for actual data, a very satisfactory adjustment is achieved without destroying the physical information contained in the data.

1. Introduction

In the computation of an energy budget for a large bounded atmospheric volume, one is faced with the problem of adjusting actual data such that the calculated mass divergence in the atmospheric volume is zero. One common method of attack is to use the line-integral method, which involves the wind component normal to the boundary sides of the volume. The boundary conditions are zero vertical velocity at the bottom and top of the atmosphere, and the net vertically integrated horizontal divergence equals zero. That is,

$$\int_{P_t}^{P_b} \oint v_n dl \frac{dp}{g} = 0, \tag{1}$$

where v_n is the velocity component normal to the bounding line element dl, dp is an increment of pressure, and g is the acceleration of gravity. The inner integral represents the horizontal divergence at a given level, and the outer integral represents the vertical integration. Since we are working in pressure coordinates the vertical velocity is actually $dp/dt=\omega$. This is not necessarily equal to zero at the bottom of the atmosphere. However, under the assumption of mass balance, the boundary condition of $\omega=0$ at the surface ($P\cong 1000$ mb) is not unrealistic. In order to obtain w, the vertical velocity using height as the vertical coordinate, the approximation $\omega\cong -\rho g w$ was used.

In calculating the integral on the left-hand side of (1) with observed winds, one often finds that mass balance under the above constraints is not obtained. The procedure which has been used by several investigators, e.g., Riehl (1958), Riehl and Malkus (1958), and Palmén *et al.* (1958), has been to average the normal component of the wind around the perimeter and then qualitatively adjust the wind in a subjective but physical way in order to obtain mass balance in the entire volume. The necessary adjustment is usually applied in regions where there is a large gap between stations.

The purpose of this paper is to outline a general objective technique for smoothing actual data in a consistent manner such that the total mass divergence in a large volume of the atmosphere is zero for the adjusted normal components of the wind data. Basically, the procedure is to modify by use of a Lagrangian multiplier a least-squares smoothing of the data for each pressure level. The technique appears to give satisfactory results for complicated theoretical functions of v_n, theoretical functions with random error superimposed on the known values, and for an actual group of atmospheric soundings.

2. The objective adjustment scheme

Consider an arbitrary volume of the atmosphere A (Fig. 1) around whose perimeter G we have a small number M of sounding stations, say on the order of 10. Let us assume that the wind data for L levels at the M stations has been analyzed and we have available estimates of v_n at each station for each pressure level. The problem is to derive an objective scheme for adjusting the v_n data such that the total mass divergence is a preset value α. In practice, the value of α will be zero, but no loss of generality occurs by permitting α to be arbitrary but specified.

The objective scheme is based on only two assumptions: 1) the investigator has firm physical grounds for specifying α, and 2) the data v_n at each pressure level l may be adequately represented by a polynomial P_n which is of degree $N-1$ where $N\leq M$. The first assumption clearly depends on the time and space scales of the

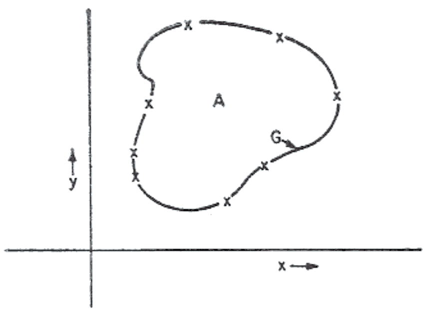

FIG. 1. The surface G encloses an arbitrary volume A. The crosses indicate the location of M upper-air sounding stations from which are available data on v_n at L pressure levels.

atmospheric phenomena which occupy the region A. The second assumption is less restrictive.

Aside from the ease of computing polynomials on a digital computer and because in classical analysis the remainder term in a Taylor series may be expressed in closed form, there is an additional theoretical advantage for choosing the class $\{P_n\}$ for adjusting the wind data. The above advantages of this class would be for naught if there were no analytic basis on which we could achieve arbitrarily high accuracy with polynomials. We assume the reader is familiar with the result (Courant and Hilbert, 1963) that the set of functions $\{P_n\}$ is complete over any interval $[a,b]$, i.e., for any piecewise continuous function $f(x)$ given any $\epsilon > 0$, there exists an n and coefficients $a_0, a_1, \cdots, a_n$ such that

$$\int_a^b \left[f(x) \quad \sum_{i=1}^{n} a_i x^{i-1} \right]^2 < \epsilon. \qquad (2)$$

Since sines and cosines also form a complete set, there is a result analogous for them. In this regard, the reader may substitute the class of Fourier functions for $\{P_n\}$ if their cyclic nature has special advantage. The result (2) assures us that we can achieve arbitrarily good least-squares approximations using linear combinations of polynomials.

Let us number the Stations $1(1)M$ and denote the distance between Station 1 and Station j as x_j where the distance has been normalized by G, i.e., the distance $x_j \leq 1$. Let V_{jl} denote v_n, i.e., the normal component of the horizontal wind for Station j and pressure level l. If ${}_lP_n$ is an appropriate polynomial for level l in the form

$${}_lP_n(x_j) = \sum_{i=1}^{n} a_{il} x_j^{i-1}, \qquad (3)$$

and we wish to find a_{il} in the least-squares sense under the constraint that the total mass divergence α be as specified, then we must minimize a function S, where

$$S(a_{il},\lambda) = \sum_{l=1}^{L} \sum_{j=1}^{M} (V_{jl} - {}_lP_n)^2 + \lambda \left[\sum_{l=1}^{L} WP_l \sum_{j=1}^{M} WL_j \, {}_lP_n - g\alpha \right], \qquad (4)$$

where λ is the Lagrangian multiplier. Golub (1965) discusses the solution of least-squares problems with constraints. The brackets, which contain the coefficient of λ, are a finite representation of (1) since we have data only at discrete points. WL_j are the integration weights for the integration around G, and WP_l are the integration weights for the integration over pressure. In practice, two applications of the repeated trapezoidal rule will suffice as a quadrature formula.

As indicated, S is a function only of a_{il} and λ. If we define the sum NL as

$$NL = \sum_{l=1}^{L} n(l), \qquad (5)$$

then there are NL a_{il}'s to determine and one λ. S is minimized by differentiating (4) $NL+1$ times. This yields the linear set of $NL+1$ equations:

$$2\sum_{j=1}^{M} V_{jl} x_j^{k} = 2\sum_{i=1}^{n(l)} a_{il} \sum_{j=1}^{M} x_j^{i-1} x_j^{k} + \lambda WP_l \sum_{j=1}^{M} WL_j x_j^{k}, \quad \begin{cases} k = 1(1)n(l), \\ l = 1(1)L, \end{cases} \qquad (6)$$

$$\sum_{l=1}^{L} WP_l \sum_{i=1}^{n(l)} a_{il} \sum_{j=1}^{M} WL_j x_j^{i} = \alpha g. \qquad (7)$$

In a matrix form (6) and (7) can be written

$$\mathbf{AU} = \mathbf{Q}, \qquad (8)$$

where $\mathbf{U}^T = [a_{11}, a_{21}, \cdots, a_{n(1),1}, a_{12}, \cdots, a_{n(L),L}, \lambda]$ (read $\mathbf{U}^T$ as $\mathbf{U}$ transpose). $\mathbf{Q}$ has as elements the left-hand side of (6) and αg. The coefficient matrix $\mathbf{A}$ is the symmetric matrix

$$\mathbf{A} = \begin{bmatrix} B_1R_1 \cdots & & \Gamma_1 \\ R_2B_2R_2 & & \Gamma_2 \\ \vdots & & \\ & R_LB_L & \Gamma_L \\ \Gamma_1^T\Gamma_2^T & \cdots & \Gamma_L^T\, 0 \end{bmatrix} \qquad (9)$$

where $\mathbf{B}_l$ is a square matrix, $\mathbf{R}_l$ a square null matrix, $\mathbf{\Gamma}_l$ a column vector, and $\mathbf{B}_l$, $\mathbf{R}_l$ and $\mathbf{\Gamma}_l$ are of order $n(l)$. $\mathbf{B}_l$ is symmetric and has the form

$$\mathbf{B}_l = 2\begin{bmatrix} (x)^0 & (x)^1 \cdots (x)^{n(l)-1} \\ (x)^1 & (x)^2 \\ \cdot & \\ (x)^{n(l)-1} & \cdots (x)^{2n(l)-2} \end{bmatrix}$$

where the notation $(x)^r$ means

$$(x)^r = \sum_{j=1}^{M} x_j^{r}.$$

Any available matrix subroutine may be used to solve (9) and determine $\mathbf{U}$.

It is pertinent to discuss the choice of n for each level. The most appropriate least-squares polynomial approxi-

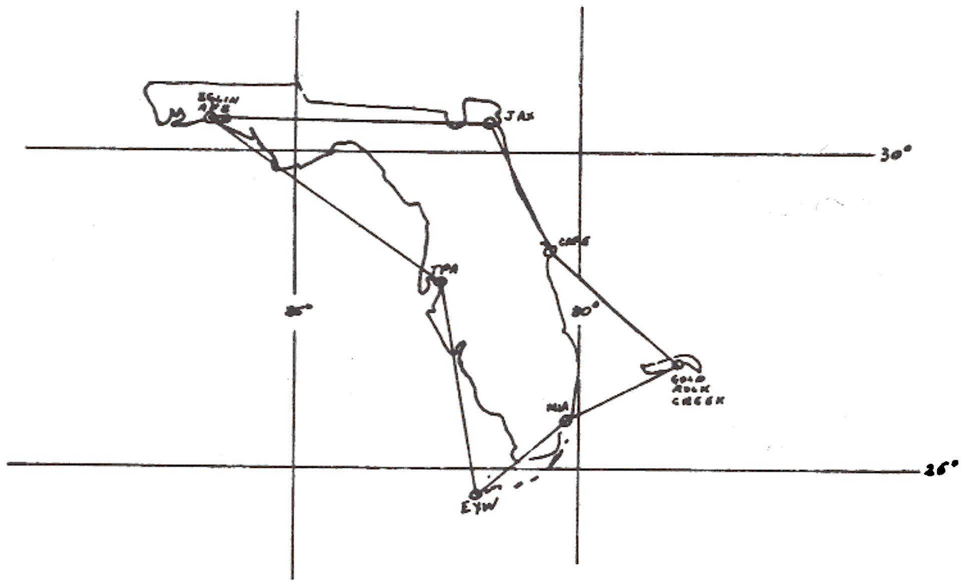

FIG. 2. Outline of the State of Florida and adjacent region. The irregular polygon represents the cross-sectional area of the volume considered. The circles represent the stations used in the analysis.

mation to a function must have two characteristics: 1) it must be of sufficiently high degree so that the approximating polynomial provides a good approximation to the real function, and 2) it must not be of such a high degree that it fits the observed data too closely in the sense that the noise or errors in the observed data are retained in the least-squares approximation. If the least-squares approximation has these two properties, then it may be said to smooth the observed data in the sense that the information available on the true function in the observed data is retained but the noise has been smoothed out.

In the case of a limited number of wind soundings, we are reluctant to apply "strong" smoothing since we fear that a physically significant portion of the information contained in the data will be lost. As we shall see in the examples presented, it is possible to make small adjustments in the wind profiles *and* obtain zero mass divergence by choosing a sufficiently high degree polynomial.

In some energy budget studies it is desirable to regard certain stations and /or levels to be more reliable (and, therefore, unalterable) than other regions in the atmospheric volume. This may be particularly true for the lowest levels in the atmosphere (S. Hastenrath, personal communication). The objective technique, outlined above, is easily adjusted to allow for these additional constraints. If the data at a particular level are regarded as unalterable, we simply choose $n(l)$ for that level to be M. In this case the fitted curve must go through each data point at that level. If the data for a particular station, say No. 3, are regarded as unalterable, then we replace ${}_lP_n(x_j)$ in (3) with

$$ {}_lP_n(x_j)=\sum_{i=1}^{n} a_{il}(x_j - x_3)^{i-1}. \qquad (12) $$

3. Theoretical examples

The objective technique will be illustrated using several examples including a few real data adjustments. First, we shall apply the technique to known theoretical wind profiles to test the concept. These specific profiles will be nondivergent except for superimposed random error.

Case A1. Consider a quadratic polynomial representation in the horizontal with a sinusoidal vertical distribution

$$ V_{jl}=(5-8X_j+8X_j^2)\sin[2\pi(\varphi_l-\varphi_L)/(\varphi_1-\varphi_L)], \quad (13) $$

where φ_l is the lth pressure level, φ_L the highest pressure level, and φ_1 the lowest level. Since (13) is nondivergent and quadratic at a particular level, the objective technique is checked by using (13) with $n(l)=3$, $l=1(1)L$. No alteration in V_{jl} occurs and the numerical technique is checked for errors.

Case A2. Consider (13) plus a random error with maximum amplitude of $\pm 0.10 V_{jl}$ from a boxcar population. Then we may define two standard errors σ: first, the standard deviation σ_E associated with the difference between the data as in (13) and the data plus error; second, the standard deviation σ_F associated with difference between the data in (13) and the least-squares obtained value. As expected, the computer results show $\sigma_F<\sigma_E$, which indicates the ability of the technique to reduce the noise in the data with random error.

Case B. Consider a nonpolynomial horizontal distribution and the same vertical distribution. Let

$$ V_{jl}=10\sin(\pi x_j+\pi/2)\sin[2\pi(\varphi-\varphi_L)/(\varphi_1-\varphi_L)]. \quad (14) $$

Again we superimpose $\pm 10\%$ error from a boxcar distribution using a random number generator. When $n=4$,

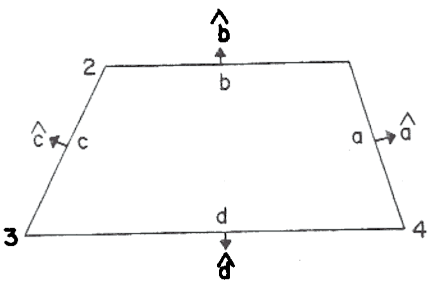

FIG. 3. Quadrilateral with Stations 1–4 at the vertices, sides a, b, c, d and unit vectors $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$.

i.e., the degree of polynomial is 3 for every level, $\sigma_F<\sigma_B$ even though V_{jl} is not polynomial in character. However, a very adequate approximation to (14) is obtained. Cases A1, A2, and B are not described in detail since we felt that atmospheric scientists would be more interested in actual application to real atmospheric data. These follow in the examples.

4. Normal wind components

Before presenting the results of the actual data cases it is pertinent to present the volume considered and the method of obtaining the wind component normal to the boundary.

The cross-sectional area of the volume is outlined in Fig. 2, with the names of the stations used in the calculations. As can be seen in Fig. 2, the area encloses most of the State of Florida and has a cross-sectional area of 1.9×10^{15} cm^2.

The normal wind components to the sides of the volume were obtained in a fashion described by Franceschini (1961). Here one assumes a linear variation of the normal wind component between stations. To illustrate the procedure consider a quadrilateral with sides of length a, b, c, d and unit vectors $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ pointing out of the volume (Fig. 3). Consider also four stations labeled 1–4 at the vertices of the quadrilateral. For the above example:

$$\oint v_n dl = \int_1^2 v_n dl + \int_2^3 v_n dl + \int_3^4 v_n dl + \int_4^1 v_n dl,$$

$$\int_1^2 \mathbf{V}\cdot\hat{\mathbf{b}}dl + \int_2^3 \mathbf{V}\cdot\hat{\mathbf{c}}dl + \int_3^4 \mathbf{V}\cdot\hat{\mathbf{d}}dl + \int_4^1 \mathbf{V}\cdot\hat{\mathbf{a}}dl,$$

$$= (\mathbf{V}_1\cdot\hat{\mathbf{b}}+\mathbf{V}_2\cdot\hat{\mathbf{b}})b/2+(\mathbf{V}_2\cdot\hat{\mathbf{c}}+\mathbf{V}_3\cdot\hat{\mathbf{c}})c/2 + (\mathbf{V}_3\cdot\hat{\mathbf{d}}+\mathbf{V}_4\cdot\hat{\mathbf{d}})d/2+(\mathbf{V}_4\cdot\hat{\mathbf{a}}+\mathbf{V}_1\cdot\hat{\mathbf{a}})a/2. \quad (15)$$

Let $b/2=B$, $c/2=C$, $d/2=D$, and $a/2=A$. Thus, after rearranging (15) becomes

$$\oint v_n dl = (\mathbf{V}_1\cdot\hat{\mathbf{b}}B+\mathbf{V}_1\cdot\hat{\mathbf{a}}A)+(\mathbf{V}_2\cdot\hat{\mathbf{b}}B+\mathbf{V}_2\cdot\hat{\mathbf{c}}C) + (\mathbf{V}_3\cdot\hat{\mathbf{c}}C+\mathbf{V}_3\cdot\hat{\mathbf{d}}D)+(\mathbf{V}_4\cdot\hat{\mathbf{d}}D+\mathbf{V}_4\cdot\hat{\mathbf{a}}A). \quad (16)$$

Now since the dot product commutes,

$$\mathbf{V}_1\cdot\hat{\mathbf{b}}B+\mathbf{V}_1\cdot\hat{\mathbf{a}}A=\mathbf{V}_1\cdot(\hat{\mathbf{b}}B+\hat{\mathbf{a}}A)=\mathbf{V}_1\cdot\hat{\mathbf{n}}N, \quad (17)$$

where $\hat{\mathbf{n}}$ is the unit normal to a line joining the midpoints of sides a and b and having magnitude N. Thus, the normal component for Station 1 is obtained, $V_{1n}=\mathbf{V}_1\cdot\mathbf{n}$, and the normal components for Stations 2 through 4 are obtained in the same manner.

5. Actual data cases

The actual atmospheric soundings selected for study were the mean July winds for the period 1957–1965 and for 0000 and 1200 GMT. The 1965 data for Eglin AFB were missing; nevertheless, the means based on eight years of data appear to be consistent with the other stations. The discussion will be in terms of the horizontal divergence and associated vertical motions rather than in terms of normal wind components. The computations were made for ten levels starting at 1000 mb and going to 100 mb in 100-mb steps.

Case C. In this case the 0000 GMT data were fitted using a 5th degree polynomial at all levels, i.e., $n(l)=5$, $l=1(1)L$. As with the theoretical wind distributions mentioned previously, $\sigma_F<\sigma_B$. Fig. 4 shows the vertical distribution of horizontal divergence both for the actual wind and the fitted wind distribution. As can be

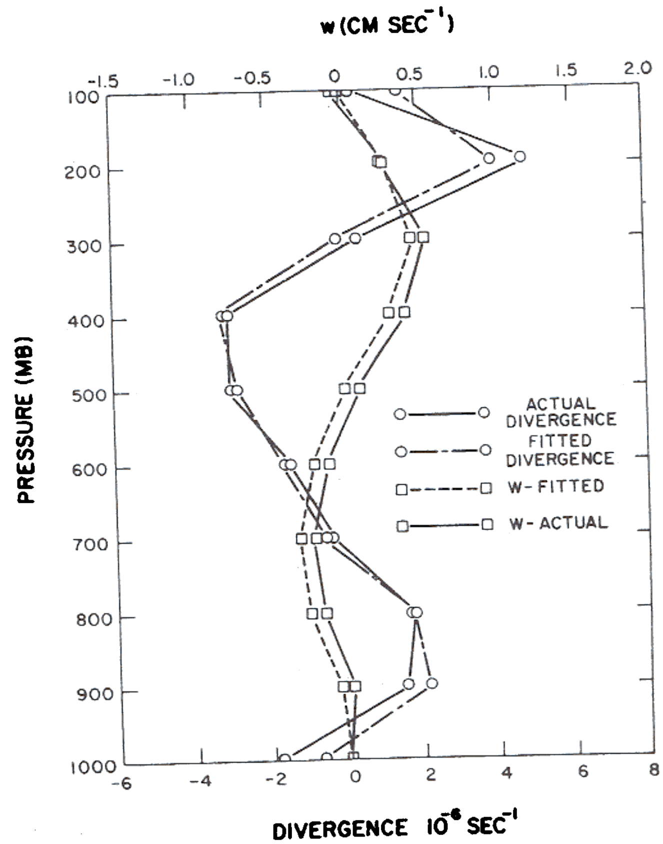

FIG. 4. Case C. The vertical distribution of divergence and vertical motion for the adjusted and unadjusted data using mean July conditions at 0000 GMT, 1957–1965. A 5th degree polynomial was fitted at all levels.

observed, the adjustment is slight at all levels but is sufficient for mass balance. The computed pressure change for the actual wind data is 0.1 mb hr^{-1} and for the fitted data is approximately 10 orders of magnitude smaller. The associated pattern of vertical motion was calculated from $\nabla \cdot \mathbf{V} = 0$ and is presented in Fig. 4. This pattern shows sinking motion below 500 mb and rising motion from 500–100 mb and is consistent with the vertical distribution of divergence.

Case C1. This case is a fit to the same data as in Case C except that a degree polynomial of $n-1$ is fitted to the levels from 1000–500 mb and a lower degree polynomial from 400–100 mb. That is, this is an attempt to make the largest adjustments in the upper levels, where the uncertainties in the winds are considered to be the greatest. Fig. 5 shows the vertical distribution of divergence and vertical motion. A comparison of Figs. 4 and 5 shows that the general features are the same. However, the fit in the lowest levels for Case C1 are very close to the values of divergence obtained from the actual wind data, while the majority of the adjustment is made at 400 mb and higher levels. The greatest differences show up in the adjustments of the normal wind component. When fitting with the lower order polynomial, the tendency is for the adjustments at individual stations at a given level to alternate between more inflow and more outflow. On the other hand, when the data is fitted with the highest order polynomial that one can use, the tendency is for a small but systematic adjustment at a given level. That is, one gets either more inflow or more outflow. Table 1 illustrates this for selected levels.

Since, when one fits the data with a degree polynomial of $N-1$, the adjustment is small but systematic at each level, it was decided to fit all ten levels in this fashion. This, in effect, allows the constraint to spread the adjustment systematically through all levels. The next case illustrates the results.

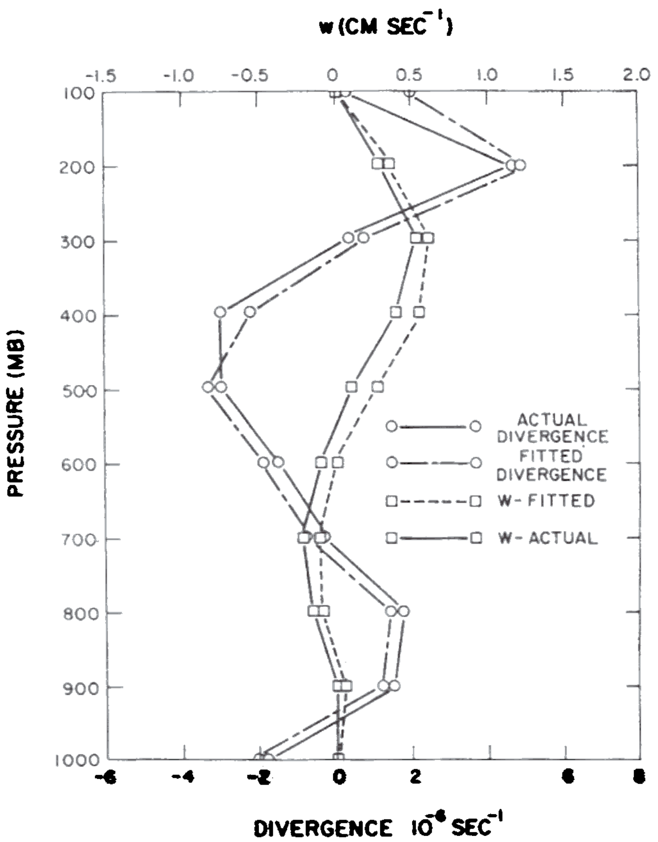

FIG. 5. Case C1. The vertical distribution of divergence and vertical motion for the adjusted and unadjusted data using mean July conditions at 0000 GMT, 1957–1965. A 6th degree polynomial was fitted from 1000–500 mb; a 5th degree polynomial from 400–100 mb.

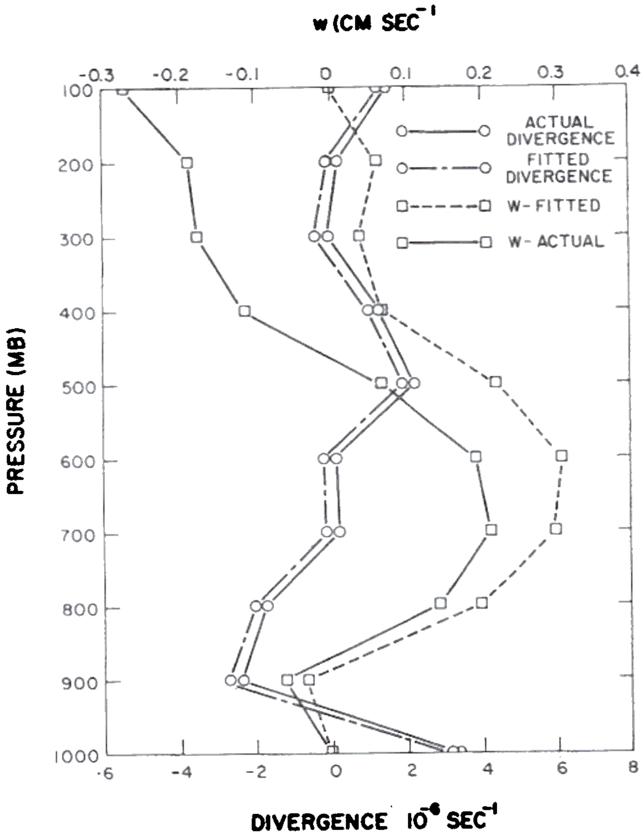

FIG. 6. Case D. The vertical distribution of divergence and vertical motion for the adjusted and unadjusted data using mean July conditions at 1200 GMT, 1957–1965. A 6th degree polynomial was fitted at all levels.

TABLE 1. Increases in inflow (negative values) and outflow (positive values) due to adjustments of the normal wind component at selected levels using different order polynomials. Values in m sec^{-1}.

Station	800 mb	800 mb	1000 mb	1000 mb
1	−0.19	−0.04	−0.14	−0.02
2	−0.02	−0.05	0.13	−0.02
3	−0.40	−0.02	−0.52	−0.01
4	0.45	−0.06	0.86	−0.03
5	−1.21	−0.02	−1.68	−0.01
6	0.78	−0.03	1.24	−0.02
7	−0.28	−0.02	−0.33	−0.01
	5th order polynomial	6th order polynomial	5th order polynomial	6th order polynomial

Case D. This case is for the mean July 1200 GMT observations. An examination of Fig. 6 shows the vertical distribution of divergence with height for both the actual winds and the fitted winds. As can be seen, the adjustment is slight but systematic at all levels. The integrated mass divergence for the unadjusted winds is about 1 mb hr^{-1} in magnitude. The corresponding value for the adjusted values is about 10 orders of magnitude smaller. The most dramatic difference shows up in the vertical motion pattern. At 100 mb the vertical motion of the unadjusted divergence is about -0.3 cm sec^{-1}, while the adjusted values meet the boundary condition of zero. At 500 mb the fitted value of vertical motion differs by a factor of two from the unadjusted value. This difference can obviously be of significance when computing vertical fluxes.

6. Conclusion

A numerical technique has been presented which allows one to arrive at mass balance using a simple least-squares fit with constraints. This procedure does not destroy the physical information contained in the actual data since the adjustments made are slight.

The choice of the degree of polynomial used to fit the data is left to the discretion of the user. Obviously, the choice will be dictated by the type of data and the confidence one has in its reliability.

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