The Influence of Bottom Topography on Baroclinic Transports

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ABSTRACT

A new and reduced set of governing equations is proposed for the modeling of baroclinic motions in a two-layer system with variable bottom topography. The reduction of the equations, which eliminates all barotropic motions, is based on the assumption of almost vertically compensated transports. The resulting equations differ somewhat from those obtained with a rigid-lid approximation. The limitations are topography variability only on horizontal scales greater than the wavelength and wavelengths shorter than the barotropic radius of deformation. These are not critical in many problems. Numerical solutions to the reduced equations are shown to be close to those obtained from the primitive equations, in one- and two-dimensional cases. In view of their relative simplicity, the new governing equations have also been applied to the analytical study of coastal upwelling in the presence of variable topography. It is shown then that the presence of a canyon enhances coastal upwelling.

The success of the reduced equations proposed here resides in the fact that, if there are no externally forced barotropic motions, those formed by the interaction of baroclinic motions over variable topography are negligible. Because of the elimination of all barotropic motions, including barotropic planetary waves and barotropic shelf waves, it is concluded that the reduced equations will be best applied to near-shore regions, fjords, small lakes and seas.

1. Introduction

A two-layer fluid system possesses two gravity wave modes, which correspond to two degrees of freedom (surface and interface displacements). These two modes are very distinct: 1) the barotropic mode, which corresponds to in-phase displacements of the surface and interface and to almost shearless motions of the water column, has a phase speed equal to \((gH)^{1/2}\), where \(g\) is gravity and \(H = H_1 + H_2\) is the total depth of the system; and 2) the baroclinic mode, which corresponds to out-of-phase displacements of the surface and interface and to shear motions of the water column, has a phase speed equal to \(\left[(\rho_2 - \rho_1)\rho_2(H_1H_2/H)\right]^{1/2}\), where \(\rho_1\) and \(\rho_2\) are the densities of the top and bottom layers, respectively. Because those two densities are generally close \([(\rho_2 - \rho_1)/\rho_2 \ll 1]\), the baroclinic phase speed is usually much less than the barotropic phase speed (typically, by a factor of 30 for oceanic systems). In two dimensions, in a rotating frame and with bottom topography, another mode, the topographic Rossby wave, is present. This mode is dispersive (its phase speed depends upon the wavenumber) but, in all practical instances, propagates very slowly.

In many geophysical situations, the two interesting modes are the baroclinic mode, which carries up-
other mode. It is this process which accounts for the production of internal tides along the shelf break (Zeilon, 1912; Rattray, 1960) and of internal waves in fjords (Stigebrandt, 1976, 1980). However, the coupling between the two modes is important only in the presence of abrupt topographic features (shelf break or fjord sill) and furthermore is much less efficient in transferring energy from an internal mode to the surface mode than vice versa (see Appendix).

The physical reason behind this asymmetry lies in the different properties of the two modes. To the order $\epsilon = (\rho_2 - \rho_1)/\rho_2 \ll 1$, baroclinic motions have no mean transport. Thus the requirement of transport continuity in the upper layer automatically leads to continuity of transport in the lower layer to the order $\epsilon$. Over variable topography, the remaining portion of transport cannot be balanced by baroclinic motions alone, and $O(\epsilon)$ barotropic motions and $O(\epsilon)$ barotropic vertical displacements of the interface and surface are generated. On the other hand, to the same order $\epsilon$, barotropic motions are shearless. Over variable topography, continuity of transport in the upper layer leads to $O(1)$ variations of transport in the lower layer (matched velocities but unmatched transports). The resulting lower-layer convergence-divergence field forces $O(1)$ baroclinic velocities and $O(\epsilon^{1/2})$ baroclinic vertical displacements of the interface. Therefore, barotropic waves are capable of generating important baroclinic motions, but the converse is not true. In systems where barotropic motions are not externally forced (and many geophysical systems fall into this category), the process by which a baroclinic wave alters itself by transferring energy to barotropic and back to baroclinic motions is only on the order of $\epsilon \times \epsilon^{-1/2} = \epsilon^{1/2} \ll 1$ and is generally inefficient.

In the literature, one finds studies of wave properties over particular topographies (Kajiura, 1974; Allen, 1975; Kawabe, 1982) and of the generation of internal waves from surface waves over variable topography (Baines, 1974; Sandstrom, 1976; Leblond and Mysak, 1978; Chao, 1980; etc.). In the later works, the emphasis is on internal-tide generation and energy transfer to baroclinic motions. Complementary to these studies which focus on the behavior of such internal waves as they are distorted, reflected and transmitted over topographic features (Baines, 1971; Chapman and Hendershot, 1981; Chuang and Wang, 1981), however, eliminate barotropic modes altogether by either assuming infinite depth or a rigid lid coupled with no rotational effects or no variation in the direction of the topography. The present work is intended to fill the gap between these two sets of studies by demonstrating that one can indeed study baroclinic motions in isolation under certain conditions without having to be limited by particular topographies or assumptions such as the rigid lid.

Using a rigid-lid approximation, Allen (1975) presented a detailed study of the interaction between the barotropic shelf wave and the internal Kelvin wave over a topography typical of a continental shelf break. The strength of the coupling critically depends upon the ratio of the internal radius of deformation over the topographic horizontal length scale. Although the two modes cannot be uncoupled, the feedback into one mode via the generation of the other is proportional to the third power of that ratio. Coupling is thus negligible as long as the topography varies slowly over one internal radius of deformation. Rather than discussing coupling phenomena, the attention of the present study is devoted to the behavior of the baroclinic signal when coupling effects can be neglected.

The technique for eliminating the barotropic mode (Section 2) is based upon the smallness of $\epsilon$, and the fact that baroclinic motions are characterized by very small vertically averaged motions. Looking for such solutions to the governing equations is made possible under reasonable restrictions, therefore proving that, under some restrictions, the elimination of the barotropic mode can be performed. Since the conditions are barely restrictive (Section 3), it is concluded that, indeed, the process by which a baroclinic wave alters itself through intermediate barotropic motions is generally negligible. Only the modulation of the mode with variations of the topography needs to be retained.

Advantages of the proposed reduced governing equations are numerous. The equations are fewer and simpler, and therefore easier to implement on the computer or even to solve analytically. Computer-time savings (not counting time savings for program development) are considerable, and the accuracy is high (Section 4). The simplicity of the proposed equations opens the door to new analytical solutions of problems hitherto thought as untractable by analytical methods. To list just a few of oceanic relevance: coastal upwelling over canyons and/or ridges (Section 5), baroclinic flow through straits, baroclinic circulation in an enclosed sea or gulf, and internal waves in fjords and lakes.

However, the equations proposed here have their inherent limitations. Since the dynamical simplification is based on the assumption of an almost vertically-compensated flow, the barotropic planetary Rossby wave that exists on a $\beta$-plane is filtered out along with the barotropic waves, in spite of its small speed of propagation. Therefore, this set of equations would not be adequate for the study of large-scale open ocean dynamics. Their applicability ought thus to be limited to near-shore regions, fjords, small lakes and seas.

2. Derivation of reduced equations

The model which will be considered in the subsequent analysis consists of a two-layer system on a
The goal is to remove the barotropic mode and focus exclusively on the baroclinic, it can be anticipated that \( \lambda_u, \lambda_v, \) and \( \lambda_h \) are still close to 1. So, taking advantage of the smallness of \( \epsilon = (\rho_2 - \rho_1)/\rho_2 \) and of the small mean baroclinic transport, one assumes

\[
\begin{align*}
    u_1 &= -(1 + \epsilon u) u_1, \\
    u_2 &= -(1 + \epsilon u) u_2
\end{align*}
\]

where \( \mu(x, y, t) \) and \( \nu(x, y, t) \) are unknown dimensionless functions on the order of unity. Elimination of \( \partial h/\partial x \) between (1) and (3), and of \( \partial h/\partial y \) between (2) and (4) yields

\[
\begin{align*}
    \frac{\partial}{\partial t} \left( H \frac{\partial}{\partial x} (\mu u_1) + f \frac{\partial}{\partial y} (\nu u_1) \right) &= -\epsilon g H \frac{\partial}{\partial x} (\mu u_1) + \epsilon g H \frac{\partial}{\partial y} (\nu u_1), \\
    \frac{\partial}{\partial t} \left( H \frac{\partial}{\partial y} (\mu u_2) + f \frac{\partial}{\partial x} (\nu u_2) \right) &= -\epsilon g H \frac{\partial}{\partial y} (\mu u_2) + \epsilon g H \frac{\partial}{\partial x} (\nu u_2).
\end{align*}
\]

Once \( \mu \) and \( \nu \) are known, the above two equations will form with the continuity equation (5) a system of three equations for the three upper-layer variables \( (u_1, v_1, h_1) \). To determine \( \mu \) and \( \nu \) one first eliminates \( \partial u_1/\partial t \) between (1) and (3), and \( \partial v_1/\partial t \) between (2) and (4), and then eliminates \( h_1 \) and \( h_2 \) by use of the continuity equations (5) and (6). The results are

\[
\begin{align*}
    &g H \left[ \frac{\partial \mu}{\partial x} (\mu u_1) + f \frac{\partial \nu}{\partial y} (\nu u_1) \right] + g H \left[ \frac{\partial^2 \mu}{\partial x^2} + \frac{\partial^2 \nu}{\partial y^2} \right] \\
    &= \frac{\partial^2}{\partial t^2} (\mu u_1) - f \frac{\partial}{\partial y} (\nu u_1) + \frac{\partial}{\partial t} \left( \frac{\partial}{\partial y} (\nu u_1) \right), \\
    &g H \left[ \frac{\partial^2 \mu}{\partial y^2} (\mu u_1) + \frac{\partial}{\partial x} (\nu u_1) \right] + g H \left[ \frac{\partial^2 \mu}{\partial y^2} + \frac{\partial^2 \nu}{\partial x^2} \right] \\
    &= \frac{\partial^2}{\partial t^2} (\nu u_1) + f \frac{\partial}{\partial x} (\mu u_1) + \frac{\partial}{\partial t} \left( \frac{\partial}{\partial x} (\mu u_1) \right).
\end{align*}
\]

If the baroclinic motions are characterized by an angular frequency \( \omega \) and wavenumber \( k \), the first two terms of the right-hand sides are negligible and can be dropped as long as

\[
\omega^2 \ll g H k^2
\]

and

\[
\omega f \ll g H k^2.
\]

It will be shown in the next section that these two requirements can be summarized by the assumption of short wavelength compared to the external (barotropic) radius of deformation. This condition is not very restrictive, for length scales of most baroclinic waves are only on the order of the internal (baroclinic) radius of deformation.
It is worth noting that it is at this precise step that the filtering of barotropic motions is performed. Indeed, the elimination of the time derivatives renders the above equations diagnostic, thus reducing the number of possible wave modes in the system of equations.

Furthermore, if the local wind stress is null, one can also drop the last terms of the right-hand sides. If the local wind stress does not vanish, a barotropic signal is present, and this component is likely to generate an important baroclinic signal over the topographic variations. The framework proposed here is thus valid only under that restriction. However, a notable exception is the coastal region within a few baroclinic radii of deformation where the offshore barotropic transport is virtually nil (Section 5). For this reason, wind stress terms are kept in the baroclinic equations.

If one further assumes that the topography is varying smoothly over one wavelength, i.e., if
\[ \frac{1}{H_2} |\nabla H_2| \ll k, \]
Eqs. (11) and (12) can be drastically reduced to
\[ \mu = v = - \frac{H_2(x, y)}{H_1 + H_2(x, y)} \]  
(13)
This result implies that, as before, \( \lambda_\nu = \lambda_\nu = 1 - (\epsilon H_2)/(H_1 + H_2) + O(\epsilon^2) \). Therefore, despite the topography variations, every aspect of the baroclinic motions is unchanged, to leading and \( \epsilon \) orders, from the case of a flat bottom. The baroclinic mode is only modulated by the bottom topography. The final equations can be summarized as
\[ \frac{\partial u_1}{\partial t} - f v_1 = -c^2 \frac{\partial h_1}{\partial x} + \frac{c^2}{\epsilon g H_1 H_2} \]  
\[ \frac{\partial v_1}{\partial t} + f u_1 = -c^2 \frac{\partial h_1}{\partial y} + \frac{c^2}{\epsilon g H_1 H_2} \]  
\[ \frac{\partial h_1}{\partial t} = - \frac{\partial u_1}{\partial x} - \frac{\partial v_1}{\partial y} \]  
(14) (15) (16)
which result from the elimination of \( \mu \) and \( v \) in (9) and (10) by use of (13). The coefficient \( c \) is defined by
\[ c^2 = \epsilon g \frac{H_1 H_2}{H_1 + H_2} \left[ 1 + \epsilon \frac{H_2}{(H_1 + H_2)^2} \right] \]  
(17)
and is recognized as the local phase speed of the baroclinic mode. Once Eqs. (14)–(16) are solved for the upper layer, the bottom layer variables can be computed by
\[ (u_2, v_2, h_2 - H_2) = -\lambda (u_1, v_1, h_1 - H_1) \]  
(18)
where \( \lambda = 1 - (\epsilon H_2)/(H_1 + H_2) \). This system of equations can easily be reduced to the one-dimensional case (no Coriolis force and no \( v \) dependence).

The rigid-lid approximation, which has been widely used in geophysical modeling to filter out the fast barotropic signal, leads to slight differences from the system of equations proposed here. Under that assumption, the surface level is forced to remain constant while a "lid" pressure field is allowed to restore the lost degree of freedom. Although this process effectively removes barotropic gravity waves, it leads to another physical system with different dynamical characteristics. As a result, the present free-surface formalism may be an improvement. Mathematically, the rigid-lid approximation leads to the same equations [(14)–(16)] but where \( c^2 \) is now given by
\[ c^2 = \epsilon g \frac{H_1 H_2}{H_1 + H_2} \left( 1 + \epsilon \frac{H_2}{H_1 + H_2} \right) \]  

and where the bottom-layer variables are found exactly equal and opposite to the top-layer variables. In summary, the rigid-lid approximation leads to a system of equations identical at the leading order but different at the first order in \( \epsilon \). In two dimensions and with rotation, the rigid-lid approximation, however, has the advantage of retaining the topographic/planetary Rossby wave and its coupling with the baroclinic motions.

Finally, it is important to mention that the above mathematical derivation has been rendered possible by the independence of the leading term of \( \lambda \) (minus unity) with respect to the space variables. To use the same formalism in an attempt to remove all baroclinic signals and to retain only the fast barotropic motions fails because the leading term of \( \lambda \) \((-H_2/H_1)\) is then not constant. Behind this mathematical argument, the physical reason for such asymmetry is the inability of barotropic motions to exist alone over variable topography, without generating a large baroclinic signal.

3. Discussion of limitations

Besides the restriction of no externally forced barotropic motion, the above formalism for the reduction of governing equations to the system (14)–(16) was based on five principal assumptions:
\[ \epsilon = (\rho_2 - \rho_1)/\rho_2 \ll 1, \]  
(19)
\[ (u_1 + u_2, v_1 + v_2, h_1 + h_2 - H_1 - H_2) = O(\epsilon), \]  
(20)
\[ \omega^2 \ll g(H_1 + H_2)k^2, \]  
(21)
\[ \omega f \ll g(H_1 + H_2)k^2, \]  
(22)
\[ \frac{1}{H_2} |\nabla H_2| \ll k. \]  
(23)

The first assumption is trivial and always met for the geophysical two-layer systems.

As mentioned earlier, the second assumption of almost vertically-compensated flow is a characteristic of all baroclinic motions and is not per se any limi-
where \( K = \frac{kc}{l} \) (dimensionless wavenumber). \( J' = \frac{\sigma}{f} \) (dimensionless frequency). \( J' \) is the two wave-number components, and \( k = (l^2 + m^2)^{1/2} \). The coefficient

\[
\gamma = \frac{(c^2) m - (c^2) l}{k^2 c^2} - \frac{\beta l}{f k^2}
\]

consists of two terms, the first one a ratio wavelength over topography length scale and the second a ratio wavelength over planetary length scale. It follows that \( \gamma \) is generally small. Dispersion curves, computed from (24), have been plotted on Fig. 1 for various values of the parameter \( \gamma \). Assumptions (21) and (22) require \( \sigma \) and \( \sigma^2 \) to much less than \( K^2/\epsilon \). Hatched areas on Fig. 1 correspond to regions where \( \sigma \) and \( \sigma^2 \) are not less than \( K^2/\epsilon \), which are thus considered as regions of non-applicability of the reduced equations.

From inspection of Fig. 1, it follows that the mixed topography-planetary baroclinic Rossby mode always falls in the region where all approximations hold, and that the gravitational baroclinic mode falls in the region of acceptability provided that

\[
x > \epsilon^{1/2} \quad \text{or} \quad k > f/(gH)^{1/2}.
\]

This final result summarizes both approximations (21) and (22). It implies that the wavelength of the baroclinic motions ought to be much shorter than the external (barotropic) radius of deformation. This condition is not very restrictive, for wavelengths of most
baroclinic waves are only on the order of the internal (baroclinic) radius of deformation. Moreover, on a non-rotating plane or in one-dimensional cases \((f = 0)\), this is no restriction at all.

4. Determination of numerical accuracy and of computer-time savings

The reduced set of equations proposed above \([(14)-(16)]\) was tested numerically by comparing its numerical solutions to the ones yielded by the primitive set \((1)-(6)\). The goal of such a test was to determine the numerical accuracy obtained from the reduced equations, as well as the computer-time savings resulting from the increased numerical stability of the finite-difference scheme. Two examples were chosen to cover both one- and two-dimensional cases.

In the first experiment, the system consisted of a one-dimensional, non-rotating, fjord-like channel closed at one end and open at the other. Calculations were started from rest. Waves were excited by imposed oscillations of the surface and interface at the open end, in such a way as to induce a baroclinic signal. The frequency of oscillations was tuned as to develop a baroclinic wavelength on the order of the channel length. The topography consisted of a large bump in the middle of the channel, reducing the bottom-layer thickness by a factor of 2 at its top. Since, under this arrangement, the topographic length scale is also about the channel length, assumption \((23)\) was pushed to its limit of validity. This was intentionally designed to evaluate a lower bound of accuracy of the reduced equations.

Physically, due to the choice of forcing, a wave enters the system at the open end, propagates throughout the channel, reflects on the other end and propagates backward, while more crests and troughs continue to enter the system at the open end. After a few periods, the incoming wave has been sufficiently deformed by the topography and the reflected wave component.

Various runs were performed to determine the maximum stability values of ratio time step/grid step \((\Delta t/\Delta x)\) of the finite-difference schemes corresponding to both the primitive and reduced sets of equations. It was found that, for the choice of parameters \((\epsilon = 0.002, H_1/H_{2\text{max}} = 0.5, H_1/H_{2\text{min}} = 1)\), the maximum ratio values leading to numerical stability were equal to 0.5 and slightly over 25.0 for the primitive and reduced equations, respectively. Therefore, although the CFL condition allowed only for an increase of the time step by a factor of 30, \(\left[H_1 + H_2\right]/\epsilon H_1 H_2 \approx 30^\frac{1}{2}\), the actual gain factor was rather 25/0.5 = 50, due probably to the simple reason that less numerical errors were accumulated by solving two equations instead of four at each time step. Moreover, for the same reason, each time step was performed twice as fast when the reduced equations were used, and therefore the total computing time was divided by a factor of 100 (50 in time-step increase, and 2 in speed of execution). Such savings are appreciable.

To evaluate the accuracy of the solution obtained from the reduced equations, shots at the solution were taken after 2.4 periods (10 000 time steps for primitive equations and 200 time steps for reduced equations). The superposition of the two results is shown in Fig. 2. To the accuracy of the computer graphic terminal, the two surface and the two interface curves are indistinguishable. Relative errors had to be determined from printed values; these were found to be only 10^{-3} for identical time steps and still a comfortable 10^{-4} for time steps separated by the factor 50. The same accuracy was found for both height and velocity fields. It is also worth recalling that these calculations were carried out for a topographic feature varying appreciably over one wavelength.

In the second numerical experiment, the system consisted of a wide, two-dimensional, rotating \((f\text{-plane})\) channel. One wall represented a coast, and the other a far remote, ineffectual boundary, located at about 25 internal radii of deformation. Off the coastal wall, the topography consisted of a linear shelf deepening offshore to a maximum total depth, beyond which the bottom was flat. (Minimum total depth at the coast and maximum offshore total depth were fixed at 3 and 5 times the upper-layer depth, respectively.) Superimposed on the shelf structure was a ridge or canyon feature of alongshore Gaussian profile and cross-shelf parabolic profile. Runs were performed with a flat bottom, a linear shelf, a linear shelf with ridge and a linear shelf with canyon. Shelf and ridge/canyon widths were also varied.

On the open ends of the channel, cyclic conditions were imposed to allow in and out propagation of coastal waves. The height and velocity fields were initialized to represent a baroclinic coastal Kelvin wave, and the time integration was performed to follow the evolution of that free wave propagating through the channel.
A first series of runs was aimed at the determination of the numerical-stability thresholds of both the primitive and reduced sets of equations. As before, it was found that, for the choice $\varepsilon = 0.002$, the maximum time step leading to stability is about 54 times greater for the reduced than for the primitive equations. This gain factor is just a little greater than 52, the actual gain factor which would be predicted from the CFL condition (based on an average total depth equal to four times the upper-layer depth). The overall computer-time savings is thus $54 \times 2 = 108$. One must bear in mind that such estimates depend on the value of $\varepsilon$ as well as on the depth ratios of the system. From the above two experiments, one can conclude that the computer time is at least divided by two times the ratio of the external radius of deformation to the internal radius of deformation.

To evaluate the accuracy of the solution obtained from the reduced equations in the two-dimensional case, a table of the relative errors of various experiments is constructed (see Table 1). Errors are evaluated for the upper-layer height and velocity fields along the coast (shallow wall). Relative amplitude errors are computed from the amplitude discrepancies between the two models at the maxima closest to the middle of the channel, while relative phase errors are taken as fractions of one wavelength between the positions of zero height and velocity disturbances, also as close to the middle of the channel as possible. From inspection of Table 1, it can be concluded that the errors are small in general. Errors on the height field are always acceptable ($<3 \times 10^{-2}$, in all cases, including sharp topographies for which assumption (23) fails), and are very small ($<5 \times 10^{-4}$) for smoother topographies, in which cases $|VH|/H \approx k$. Errors on the velocity field are generally higher (up to $4\%$ for sharp topographies), but reasonable ($<1.1 \times 10^{-2}$) for smoother topographies. From Table 1, it also becomes evident that the baroclinic model (reduced equations) works better if topographic features such as canyons and ridges are isolated, than if topographic features such as a narrow shelf are extended throughout the system. Indeed, a shelf structure allows for the existence of a topographic Rossby wave, present in the two-layer model but absent in the reduced model. Phase errors are always acceptable ($<1\%$ of a wavelength).

Comparison of the error results from these two numerical experiments reveals that the reduced set of equations is more faithful in a one-dimensional, boundary-forced case than in a two-dimensional, initial-value case. However, one should realize that, in the first experiment, any barotropic mode can hardly be maintained, for the boundary conditions require it to have a velocity node at the closed end and a height node at the other. In the second experiment, not only barotropic topographic modes can develop, but also the surface gravity mode is not constrained at either boundary. It should be concluded that boundary-value and initial-value problems can be solved with the reduced set of equations, with errors on the order of those found for the first and second numerical experiments, respectively.

5. Analytical application to coastal upwelling

The relative simplicity of the reduced set of equations established here renders some problems much more tractable by analytical methods of solution than otherwise. One such problem is the influence of topographic features on coastal upwelling events. This example is chosen to illustrate the benefit of simplicity which results from this set of equations.

Observations (Shaffer, 1976; Gostan and Guibaut, 1974; Millot and Wald, 1981; Moody et al., 1981) as well as numerical modeling (Peffley and O'Brien, 1976; Peffer and O'Brien, 1980) all show that coastal upwelling is more pronounced near the shore above

<table>
<thead>
<tr>
<th>Case</th>
<th>Height-field error</th>
<th>Phase</th>
<th>Velocity-field error</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wide shelf only</td>
<td>(3.0, 2.8, 3.2) x 10^{-3}</td>
<td></td>
<td>(2.1, 1.7, 0.9) x 10^{-4}</td>
<td></td>
</tr>
<tr>
<td>Narrow shelf only</td>
<td>(4.0, 3.8, 3.3) x 10^{-4}</td>
<td></td>
<td>(8.5, 8.2, 7.1) x 10^{-3}</td>
<td></td>
</tr>
<tr>
<td>Wide shelf/wide canyon</td>
<td>(6.2, 5.4, 3.2) x 10^{-3}</td>
<td></td>
<td>(5.7, 5.1, 3.8) x 10^{-3}</td>
<td></td>
</tr>
<tr>
<td>Wide shelf/narrow canyon</td>
<td>(2.7, 2.1, 0.7) x 10^{-4}</td>
<td></td>
<td>(3.4, 3.3, 3.5) x 10^{-3}</td>
<td></td>
</tr>
<tr>
<td>Narrow shelf/wide canyon</td>
<td>(7.5, 7.1, 6.4) x 10^{-4}</td>
<td></td>
<td>(1.0, 1.0, 1.0) x 10^{-2}</td>
<td></td>
</tr>
<tr>
<td>Narrow shelf/narrow canyon</td>
<td>(7.5, 7.1, 6.3) x 10^{-4}</td>
<td></td>
<td>(1.0, 1.0, 1.1) x 10^{-2}</td>
<td></td>
</tr>
<tr>
<td>Wide shelf/wide ridge</td>
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<td>(1.1, 1.2, 1.3) x 10^{-4}</td>
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<tr>
<td>Wide shelf/narrow ridge</td>
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<td></td>
<td>(8.7, 7.9, 6.0) x 10^{-4}</td>
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<td></td>
</tr>
</tbody>
</table>
alongshore transport and upper-layer depth fluctuations derived from (14)-(16): frequency space and fluctuating in time with a sub-inertial frequency.

According to (25), in a canyon where the bottom layer is greater and, hence, at a given time, the upwelling is deeper than in the surroundings, the baroclinic flow is greater and, hence, the amount of water pumped into the bottom layer and thus the upwelling signal ought to be stronger.

The correctness of this argument is confirmed by the analytical solution obtained from the reduced equations applied to coastal upwelling. Assuming an alongshore (y-directed) wind stress \( \tau_0 \) uniform in space and fluctuating in time with a sub-inertial frequency \( \omega \) (to study both upwelling and downwelling), one ought to solve the following equations derived from (14)-(16):

\[
\begin{align*}
\tau \partial_t \mathbf{u} &= -\mathbf{f} - \mathbf{v} \times \mathbf{u}, \\
\mathbf{f} + \mathbf{v} \times \mathbf{u} &= -c^2 \mathbf{v} + \frac{\tau \tau_0}{\rho g H_1}, \\
\tau \mathbf{h} + \mathbf{u} &+ \mathbf{v} = 0,
\end{align*}
\]

where \( u, v \) and \( h \) are the top-layer inshore transport, alongshore transport and upper-layer depth fluctuations, respectively. Although a barotropic component is present and although it plays a crucial role in taking over part of the far-offshore Ekman flux, the above baroclinic equations are applicable. Indeed, the solution they yield is spatially varying only within a few internal radii of deformation offshore, where the barotropic velocity component is null and the barotropic height variation very small (order \( \epsilon \)).

The elimination of \( u \) and \( v \) reduces the problem to a simple equation for the height field \( h \), i.e.,

\[
\begin{align*}
\nabla^2 h - \frac{\tau^2}{c^2} - \frac{2}{c} \nabla_c \cdot \nabla h - \frac{2i\epsilon}{\omega c} J(c, h) \\
+ \frac{2\tau_0}{\epsilon g H_1} \left( \frac{i\epsilon}{c} \right) &= 0,
\end{align*}
\]

where \( J \) is the Jacobian operator. After one scales \( \omega \) by \( f \), \( c \) by \( (\epsilon g H_1)^{1/2} \), \( \tau_0 \) by \( \rho g f (\epsilon g H_1)^{1/2} \), and the coordinates by \( (\epsilon g H_1)^{1/2}/f \) (short scale) and \( L \) (topography length scale), a small parameter appears:

\[
\lambda = \frac{(\epsilon g H_1)^{1/2}}{fL} \quad \text{is real and positive (} \omega < 1 \text{).}
\]

The solutions to the above equations are simple if one assumes that there are no coastal propagating waves (locally-forced upwelling):

\[
h_0 = -\frac{i}{\omega \lambda} e^{\lambda x},
\]

\[
\frac{i \epsilon}{2\omega \lambda^2 c} (1 - 3\lambda x + \lambda^2 x^2) e^{\lambda x} \\
+ \frac{1}{\omega^2 \lambda^2 c} (2 - \lambda x) e^{\lambda x} + \frac{2i \epsilon}{\omega \lambda^2 c} - 2 \frac{\epsilon}{\lambda c}.
\]

The leading order solution (26) already exhibits the enhancement of the upwelling signal at the coast over a canyon. Indeed, where the depth is greater, \( c, \lambda^{-1} \) and thus \( h_0(x = 0) \) are larger. This is a direct con-
Combining the leading- and first-order solutions along with the explicit time dependence, one notes that an upwelling-downwelling cycle in response to oscillating along-shore winds can be reproduced. Figs. 3 and 4 show the interface anomaly, respectively, at the time of maximum upwelling-favorable winds and at the time of vanishing winds after upwelling and before downwelling. The topography is identical to the linear shelf with canyon structure as used in the bi-dimensional numerical check in Section 4. These plots demonstrate the upwelling enhancement over the canyon. Since this model is linear, the downwelling signal (not shown) is identical but opposite to the upwelling signal and is, therefore, also enhanced over a canyon.

The same analytical result was also applied to a case where the topography approximates the bathygraphy along the coast of Peru. As seen in Fig. 5, the upwelling is more intense along the shore where the shelf is narrower, while the seamount has a slight, local, upwelling-enhancing effect. This compares favorably to observations (Moody et al., 1981) as well as to numerical simulation (Preller and O'Brien, 1980).

6. Conclusions

The reduced set of governing equations established here for two-layer systems has been shown to be simple, accurate and versatile. Its success resides in the fact that, if there are no externally forced barotropic motions, those formed by the interaction of baroclinic motions over variable topography are negligible.

As presented here, the reduced equations govern the variations of the upper-layer variables only. They are analogous in formulation to those of reduced-gravity and rigid-lid models, but with a variable baroclinic propagation speed locally dependent on the bottom topography. Once the solution for the upper-layer variables is known, the solution in the lower layer is determined by simple algebraic relations.

The rigorous derivation of the baroclinic reduced equations is based on the assumption of almost vertically compensated transports, for which a solution exists provided that there is no externally forced barotropic component and that a few other requirements are met. These requirements can be summarized into two limitations: 1) the topography should not vary on horizontal scales smaller than the wavelength; and 2) wavelengths should be much shorter than the barotropic radius of deformation. For many geophysical situations, these limitations are hardly severe. The model proposed here can be best applied to coastal processes, fjords, small lakes and seas, although it is definitely not applicable to coastal dynamics when a barotropic signal (surface tide or shelf wave) is present over sharp topography (shelf break) nor to open-ocean dynamics if barotropic planetary waves are important.

Numerical testing of the reduced set of equations in two typical cases has determined the accuracy and the computer-time savings that one can anticipate.
The accuracy in both wave amplitude and phase is very acceptable as long as the above requirements are met. The accuracy is particularly good for systems that can hardly maintain any barotropic signals (boundary-forced or one-dimensional). The computer-time savings are very appreciable for explicit finite-difference schemes (gain factor of about 100) owing to the combined advantages resulting from solving half as many simpler equations in which the maximum-phase speed of wave propagation has been much reduced.

The analytical application to coastal upwelling over topography presented here illustrates that the simplicity of the equations produce results easy to interpret. As an example, it is demonstrated how a canyon can act to enhance an upwelling event. Besides this particular application, the simplicity of the reduced equations permits various analytical modeling attempts, such as baroclinic flow through straits, baroclinic circulation in lakes, gulfs or small seas, etc.

Finally, this paper applies to two-layer systems exclusively. It remains to generalize the approach to system including more than two layers and, possibly, to continuous stratification.

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APPENDIX

Coupling over a Topographic Step

Here we will illustrate the coupling between barotropic and baroclinic modes in a two-layer system over a topographic step. As a result, it is demonstrated that an incident barotropic mode generates a large baroclinic component (order $\epsilon^{1/2}$), whereas an incident baroclinic mode generates only a very small barotropic component (order $\epsilon$).

The governing equations are the one-dimensional shallow-water equations of a two-layer system. Assuming a progressive wave of frequency $\omega$ and wave-number $k$, one obtains:
where \( u_1 \) and \( u_2 \) are the upper and lower transports, respectively, \( \eta_1 \) and \( \eta_2 \) the thickness variations of the top and bottom layer, respectively, and \( H_1 \) the top-layer depth at rest. The phase speed \( c = \omega / k \) is prescribed to be a solution of the dispersion relation:

\[
\frac{c^4}{g^2H_1H_2} - \frac{H_1 + H_2}{gH_1H_2} c^2 + \epsilon = 0, \tag{A4}
\]

where \( \epsilon = (\rho_2 - \rho_1)/\rho_2 \) is a measure of the density difference between the two layers. This quartic equation possesses four solutions corresponding to both barotropic and baroclinic modes either propagating toward or away from the topographic step. Note that due to the step variation in \( H_2 \), the bottom-layer depth at rest, those solutions are not the same on either side of the step.

At this point, notation must clearly differentiate between all different modes; from now on, an upper bar will refer to a barotropic mode, a tilde to a baroclinic mode, plus and minus signs will refer to propagation toward the positive and negative \( x \) direction, respectively, and primes and no primes will designate the two sides of the step.

Continuity of transports \( (u_1 \) and \( u_2 \) and of layer thicknesses \( (\eta_1 \) and \( \eta_2 \)) across the region of the topographic step yields:

**CONTINUITY OF \( \eta_1 \):**

\[
(\tilde{\eta}_1^+ + \tilde{\eta}_1^-) + (\tilde{\eta}_1^+ + \tilde{\eta}_1^-) = (\tilde{\eta}_1^+ + \tilde{\eta}_1^-), \tag{A5}
\]

**CONTINUITY OF \( u \)**

\[
\tilde{c}(\tilde{\eta}_1^+ - \tilde{\eta}_1^-) + \tilde{c}(\tilde{\eta}_1^+ - \tilde{\eta}_1^-) = \tilde{c}(\tilde{\eta}_1^+ - \tilde{\eta}_1^-) + \tilde{c}(\tilde{\eta}_1^+ - \tilde{\eta}_1^-), \tag{A6}
\]

**CONTINUITY OF \( \eta_1 + \eta_2 \)**

\[
\tilde{c}^2(\tilde{\eta}_1^+ + \tilde{\eta}_1^-) + \tilde{c}^2(\tilde{\eta}_1^+ + \tilde{\eta}_1^-) = \tilde{c}^2(\tilde{\eta}_1^+ + \tilde{\eta}_1^-) + \tilde{c}^2(\tilde{\eta}_1^+ + \tilde{\eta}_1^-), \tag{A7}
\]

**CONTINUITY OF \( u_1 + u_2 \)**

\[
\tilde{c}^2(\tilde{\eta}_1^+ - \tilde{\eta}_1^-) + \tilde{c}^2(\tilde{\eta}_1^+ - \tilde{\eta}_1^-) = \tilde{c}^2(\tilde{\eta}_1^+ - \tilde{\eta}_1^-) + \tilde{c}^2(\tilde{\eta}_1^+ - \tilde{\eta}_1^-). \tag{A8}
\]

Solving for the mode amplitudes of one side of the step in function of the amplitudes on the other side, one obtains:

\[
\eta_{1z} = \frac{1}{2} \left( \frac{c^2 - \epsilon^2}{c^2 - \epsilon^2} \right) \left[ (1 + \frac{c}{\epsilon}) \tilde{\eta}_{1z} + (1 - \frac{c}{\epsilon}) \tilde{\eta}_{1z} \right] + \frac{1}{2} \left( \frac{c^2 - \epsilon^2}{c^2 - \epsilon^2} \right) \left[ (1 + \frac{c}{\epsilon}) \tilde{\eta}_{1z} + (1 - \frac{c}{\epsilon}) \tilde{\eta}_{1z} \right]
\]

It can be shown from the above expressions that an incident barotropic wave \( (\tilde{\eta}_{1z} = 1, \tilde{\eta}_{1z} = \tilde{\eta}_{1z} = \tilde{\eta}_{1z} = 0) \) leads to the generation of a large baroclinic wave \( (\tilde{\eta}_{1z} = 1, \tilde{\eta}_{1z} = \tilde{\eta}_{1z} = \tilde{\eta}_{1z} = 0) \), and that an incident baroclinic wave \( (\tilde{\eta}_{1z} = 1, \tilde{\eta}_{1z} = \tilde{\eta}_{1z} = \tilde{\eta}_{1z} = 0) \) leads to the generation of a very small barotropic wave \( (\tilde{\eta}_{1z} = \tilde{\eta}_{1z} = \tilde{\eta}_{1z} = 0) \). And, therefore, the feedback on the baroclinic mode via generation of a barotropic mode is thus on the order of \( \epsilon \times \epsilon^{-1/2} = \epsilon^{1/2} \) and is negligible. This argument formulated here in terms of interface heights also holds in terms of velocities. In particular, an incident barotropic wave \( (\tilde{\eta}_{1z} = 1, \tilde{\eta}_{1z} = 0) \) leads to the generation of a very small barotropic \( (\tilde{\eta}_{1z} = 1, \tilde{\eta}_{1z} = 0) \), and that an incident barotropic wave \( (\tilde{\eta}_{1z} = 1, \tilde{\eta}_{1z} = 0) \) leads to the generation of a very small baroclinic \( (\tilde{\eta}_{1z} = 1, \tilde{\eta}_{1z} = 0) \). These are negligible compared to the velocities in the incident wave. Over smooth topography, the same or a weaker feedback is expected.

The physical reason behind this dissymmetric behavior in the coupling of the two modes lies in the fact that, to order \( \epsilon \), the baroclinic mean transport is null. The matching of the upper-layer transport thus automatically balances the lower-layer transport to the order \( \epsilon \), and only a weak barotropic signal on the order of \( \epsilon \) is called to close the mass-flow requirement. Yet, in the case of an incident barotropic wave, the matching of both upper and lower transports is impossible without baroclinic transports of order unity, i.e., interface variations on the order \( \epsilon^{-1/2} \).

**REFERENCES**


