THE HYPERBOLIC PROBLEM

J. J. O'Brien
Mesoscale Air-Sea Interaction Group
The Florida State University
Tallahassee, FL 32306-3041

1. INTRODUCTION

Before we proceed to realistic ocean modelling problems, it is useful to have a knowledge of finite difference schemes for simple hyperbolic partial differential equations (PDE).

2. HYPERBOLIC EQUATIONS

The general linear second-order PDE is

\[ au_{xx} + 2bu_{xy} + cu_{yy} + 2du_x + 2eu_y + f = h(x,y) \]

If \( b^2 - ac > 0 \), then (1) is hyperbolic. We can eliminate the \( u_{xy} \) term by the transformation

\[
\begin{align*}
    y - \lambda_1 x &= \xi + \eta \\
    y - \lambda_2 x &= \xi - \eta
\end{align*}
\]

where \( \lambda_1, \lambda_2 \) are the roots of \( a\lambda^2 - 2b\lambda + c = 0 \). The transformed equation is

\[ u_{\xi\xi} - u_{\eta\eta} + 2Du_\xi + 2Eu_\eta + F = H(\xi,\eta) \]

This is the canonical form of a second order hyperbolic equation. A commonly encountered hyperbolic equation is the wave equation

\[ u_{tt} - c^2 u_{xx} = g(x) \]

which can be written as the system of first order equations (if \( c \) is a constant).
\[ u_t = cu_x + v \]
\[ v_t = cu_x \]

In fluid dynamics we frequently encounter systems of hyperbolic equations. Suppose we have a system of equations for the vector
\[ u(x,t) = [u_1, u_2, \ldots, u_n]^T \]
such as
\[ \begin{align*}
\frac{\partial u_1}{\partial t} &= a_{11} \frac{\partial u_1}{\partial x} + a_{12} \frac{\partial u_2}{\partial x} + \ldots + a_{1n} \frac{\partial u_n}{\partial x} \\
\frac{\partial u_2}{\partial t} &= a_{21} \frac{\partial u_1}{\partial x} + a_{22} \frac{\partial u_2}{\partial x} + \ldots \\
& \vdots \\
\frac{\partial u_n}{\partial t} &= a_{n1} \frac{\partial u_1}{\partial x} + a_{n2} \frac{\partial u_2}{\partial x} + \ldots + a_{nn} \frac{\partial u_n}{\partial x}
\end{align*} \]

which can be written in matrix form
\[ \frac{\partial u}{\partial t} = A \frac{\partial u}{\partial x} \]

Suppose that \( A \) has \( n \) distinct real eigenvalues \( \{\lambda_i\} \); then the system is hyperbolic and there exists a nonsingular matrix, \( P \), of eigenvectors such that
\[ D = PAP^{-1} \]
is a diagonal matrix whose diagonal elements are \( \{\lambda_i\} \). Let \( w = Pu \), then
\[ w_t = (PAP^{-1}) Pu_x = Dw_x \]
or
\[ \begin{align*}
\frac{\partial w_1}{\partial t} &= \lambda_1 \frac{\partial w_1}{\partial x} \\
& \vdots \\
\frac{\partial w_n}{\partial t} &= \lambda_n \frac{\partial w_n}{\partial x}
\end{align*} \]
where each equation is a simple hyperbolic equation; thus the simplest model equation for linear hyperbolic problems is the one-dimensional advection equation

\[ u_t + Au_x = 0 \]

with initial condition \( u(x,0) = f(x) \) for \(-\infty < x < \infty\). The solution is well-known; i.e., \( u(x,t) = f(x - At) \).

3. FINITE DIFFERENCE SCHEMES FOR ONE-DIMENSIONAL HYPERBOLIC EQUATIONS

There are numerous methods for handling hyperbolic problems. Several frequently encountered schemes follow.

The model problem is

\[ \frac{\partial u}{\partial t} = -A \frac{\partial u}{\partial x} \]

where \( u(x,t) \) is real, \( A \) is a real constant; define \( a = A\Delta t/\Delta x \), \( \alpha = k\Delta x \), where \( \Delta t \) is the time step and \( \Delta x \) is the grid spacing and \( k \) is a wavenumber. The dimensionless number, \( a \), is called the Courant number.

**Method 1**

Unstable (forward time step and centered in space)

\[ u_{j+1}^{n+1} = u_j^n - a \left( \frac{u_j^{n+1} - u_j^n}{2} \right) \]

\[ G = 1 - i a \sin \alpha \]

This scheme is always unstable because \( |G| \geq 1 \). This is an important result. For a diffusive problem, forward in time and centered in space is conditionally stable, but for the hyperbolic problem it is unconditionally unstable.

**Method 2: Upstream Differencing**

\[ u_{j+1}^{n+1} = u_j^n - a \]

\[ u_j^{n+1} - u_j^n, \text{ if } a < 0 \]

\[ e^{i\alpha} - 1, \quad a < 0 \]

\[ 1 - e^{-i\alpha}, \quad a > 0 \]

\[ G = 1 - a \]

\[ G = 1 - a \]
This scheme is stable if $|a| \leq 1$. It is very popular in meteorology and oceanography because of the ease of implementation. However, this author strongly opposes the use of upstream differencing because it is a very dissipative finite difference scheme. This is demonstrated by considering the case, $a > 0$ and adding a particular zero to (4)

$$u_{j}^{n+1} = u_{j}^{n} - a(u_{j}^{n} - u_{j-1}^{n}) + \frac{a}{2}(u_{j+1}^{n} - u_{j-1}^{n})$$

The last term is obviously zero. By rearranging, we can obtain

$$u_{j}^{n+1} = u_{j}^{n} - \frac{a}{2}(u_{j+1}^{n} - u_{j-1}^{n}) + \frac{a}{2}(u_{j+1}^{n} + u_{j-1}^{n} - 2u_{j}^{n})$$

The first two terms on right are the unstable scheme; the last term is a second-order approximation to a diffusion equation! If we use upstream differencing, we are actually solving the differential equation

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = k\frac{\partial^2 u}{\partial x^2}$$

with the computational viscosity, $A\Delta x/2$, $A > 0$, and an unconditionally unstable approximation for the advection term. We have added enough artificial friction to damp the unstable part of the problem. In practice, $A\Delta x/2$ represents a value for eddy viscosity that no one would attribute physically to a large scale flow problem in meteorology or oceanography. In fact, things are humorous. There are several published papers with real diffusive terms for which the authors "learned" that the value of eddy viscosity did not effect their solution, but they had used upstream differencing for which $A\Delta x$ was at least an order of magnitude larger than their prescribed eddy viscosity! Naturally, we should not give the references here.

**Exercise:** Derive the computational viscosity for the case $a < 0$. 
Method 3: Diffusion Scheme (Friedrich’s scheme)

\[ u_j^{n+1} = \frac{1}{2}(u_j^{n+1} + u_j^n) - \frac{a}{2}(u_{j+1}^n - u_{j-1}^n) \]  \hspace{1cm} (5)

\[ G = \cos \alpha - \text{i} a \sin \alpha \]

Stable if \(|a| < 1\).

For the time derivative, if we use \((u_j^{n+1} - u_j^n)/\Delta t\) the scheme will be unstable. To avoid this problem, we replace \(u_j^n\) by the average between \(u_{j+1}^n\) and \(u_{j-1}^n\) so that the stencil becomes:

\[ \begin{array}{c}
0 \\
0 \\
0 \\
\end{array} \]

\[ |G|^2 = \cos^2 \alpha + a^2 \sin^2 \alpha : \text{if } |a| \leq 1 \text{ then } |G| \leq 1, \text{ i.e. } \frac{|A|\Delta t}{\Delta x} < 1 \]

This scheme is the first one we have seen which is staggered in time-space. As for the upstream scheme, we can consider the discretization to be from

\[ \frac{\partial u}{\partial t} = -A \frac{\partial u}{\partial x} + k \frac{\partial^2 u}{\partial x^2} \]  \hspace{1cm} (6)

which we can rewrite in the discrete form:

\[ u_j^{n+1} = u_j^n - \frac{a}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{K \Delta t}{2(\Delta x)^2}(u_{j+1}^n + u_{j-1}^n - 2u_j^n) \]

The advection part is the unstable scheme and the diffusive part is the conditionally stable forward difference. In this diffusive scheme, the artificial viscosity is \(K = (\Delta x)^2/\Delta t\).

Method 4: Leapfrog

This is one of the most popular schemes in oceanography and meteorology. We choose to use a centered-in-time, centered-in-space scheme.

\[ u_j^{n+1} = u_j^{n-1} - a(u_{j+1}^n - u_{j-1}^n) \]
The stencil is

```
 0
0--|--|--|-- 0
|    |
0
```

The amplification factor is

\[ G = g^{-1} - a \sin \alpha \]

or

\[ G = i a \sin \alpha \pm (-a^2 \sin^2 \alpha + 1)^{1/2} \]

If \( a^2 \sin^2 \alpha \leq 1 \), the radical is real and \(|G| = 1\). There are several points to notice. If \(|a| > 1\), the most unstable wave is \( \alpha = \pi/2 \) or \( k = 2\pi/\Delta x \). If we violate the stability condition, the four \( \Delta x \) wave will grow fastest. Recall in the diffusive problem that the two \( \Delta x \) wave was the most unstable.

**Definition:** The ratio, \( \Delta t/\Delta x \) is called the Courant Number. The condition \( \Delta t/\Delta x \leq 1 \) for stability is called the CFL Condition after Courant, Friedrich, and Lewy. Physically the CFL condition states that useful information must propagate less than one \( \Delta x \) in time \( \Delta t \).

**Method 5: Lax-Wendroff Scheme for the advection equation**

Consider

\[ \frac{\partial u}{\partial t} = -A \frac{\partial u}{\partial x} \]

If \( A \) is constant, the second derivative in time is

\[ u_{tt} = -Au_{xt} = A^2 u_{xx} \]

A Taylor Series expansion in time yields

\[ u^{n+1}_j = u^n_j + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} + \ldots \]

or

\[ u^{n+1}_j = u^n_j - A\Delta t \frac{\partial u}{\partial x} + \frac{\Delta t^2 A^2}{2} \frac{\partial^2 u}{\partial x^2} + O(\Delta t^3) \]
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Let \( a = \frac{A\Delta t}{\Delta x} \), then the finite difference scheme is

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\Delta x}{2}[u_{j+1}^n - u_{j-1}^n] + \frac{a^2}{2}[u_{j+1}^n - 2u_j^n + u_{j-1}^n] + O(\Delta t^3)
\]

The linear stability analysis for this scheme is instructive; let

\[
u_j^n = u_n e^{i k j \Delta x}
\]

\[
G = 1 - i a \sin k\Delta x + a^2[\cos k\Delta x - 1]
\]

\[
|G|^2 = 1 + 2a^2(\cos \theta - 1) + a^4(\cos \theta - 1)^2 + a^2 \sin^2 \theta
\]

\[
= 1 - a^2(1 - a^2)(1 - \cos \theta)^2
\]

If \( |a| < 1 \), then \( a^2(1 - a^2) \leq 1/4 \)

and \( 1 - \cos \theta)^2 \leq 4 \)

therefore \( |G|^2 \leq 1 \) if \( |a| < 1 \).

Suppose we were solving the following equation:

\[
\frac{\partial u}{\partial t} = -A \frac{\partial u}{\partial x} + K \frac{\partial^2 u}{\partial x^2}
\]

if

\[
\frac{a^2}{2} = \frac{K\Delta t}{(\Delta x)^2}
\]

then

\[
\frac{a^2\Delta t^2}{2(\Delta x)^2} = \frac{K\Delta t}{(\Delta x)^2}
\]

Thus \( K = A^2\Delta t/2 \) is the Artificial Viscosity induced by the Lax-Wendroff scheme.

Method 6: The Two-Step Lax-Wendroff Scheme

For primitive equation models, it is awkward to use the Lax-Wendroff scheme because we have to differentiate the equations in time. With non-linear problems, this leads to complicated spatial derivatives. There is a two step version of this idea which is
particularly useful for nonlinear problems.

For step one, use the diffusing time step (Method 2); for step two, use leapfrog (Method 4). It is important to realize that we do not consider the solution at odd time steps a solution of our equation.

**Step 1: Diffusing Time Step**

\[
\begin{align*}
  u_{j}^{n+1} &= \frac{1}{2}(u_{j+1}^{n} + u_{j-1}^{n}) + \frac{A\Delta t}{2Ax}(u_{j+1}^{n} - u_{j-1}^{n})
\end{align*}
\]

**Step 2: Leap-Frog**

\[
\begin{align*}
  u_{j}^{n+2} &= u_{j}^{n} + \frac{A\Delta t}{\Delta x}(u_{j+1}^{n+1} - u_{j-1}^{n+1})
\end{align*}
\]

The two-step method is best suited for 2-D problems and non-linear problems.

The stencil is

The circles are the first step; the crosses are the second step. Note we have to solve for all points at time level, \( n+1 \), before finding solution at \( (n+2)\Delta t \).

We can show that the two-step method is equivalent to the one-step method. Let us eliminate the dependence on \( (n+1) \)

\[
\begin{align*}
  u_{j}^{n+2} &= u_{j}^{n} - a\left[\frac{1}{2}u_{j+2}^{n} + \frac{1}{2}u_{j}^{n} - \frac{2}{2}(u_{j+2}^{n} - u_{j}^{n})
  - \frac{1}{2}u_{j}^{n} - \frac{1}{2}u_{j-2}^{n} + \frac{1}{2}(u_{j}^{n} - u_{j-2}^{n})\right]
\end{align*}
\]

\[
\begin{align*}
  u_{j}^{n+2} &= u_{j}^{n} - a\left[\frac{1}{2}(u_{j+2}^{n} - u_{j}^{n}) + \frac{a^{2}}{2}(u_{j+2}^{n} - 2u_{j}^{n} + u_{j-2}^{n})\right]
\end{align*}
\]

Therefore this is the same as the one-step method if \( \Delta x \) is \( (2\Delta x) \), and \( \Delta t \) is \( (2\Delta t) \).
The equations are

\[ u_{j+1}^{n+1} = \frac{1}{2}(u_j^n + u_{j+2}^n) - \frac{a}{2}(u_{j+2}^n - u_j^n) \quad j \text{(odd)} \]

\[ u_j^{n+2} = u_j^n - a(u_{j+1}^{n+1} - u_{j-1}^{n+1}) \quad j \text{(even)} \]

Observe that we have thrown away half the space points for each time level. We note that the scheme is said to be staggered in time and space.

**Method 7: Implicit Scheme**

We write:

\[ u_{j+1}^{n+1} = u_j^n - \frac{a}{2} \left( \frac{u_{j+1}^n + u_{j+1}^{n+1}}{2} - \frac{u_{j-1}^n + u_{j-1}^{n+1}}{2} \right) \]

For stability, let \( u_j^n = U_n e^{ikj\Delta x} \)

\[ U_{n+1} = U_n - \frac{a}{2} \left( \frac{2\text{isina}}{2} U_{n+1} + \frac{2\text{isina}}{2} U_n \right) \]

Then \(|G|^2 = (1 + \frac{a^2\text{sin}^2\alpha}{4})/(1 + \frac{a^2\text{sin}^2\alpha}{4}) = 1\)

This scheme is neutral and stable. In practice we never use implicit schemes for advection problems in oceanography because the equations are nonlinear.
Method 8: Quasi-Lagrangian

This idea is an important concept which is not used as frequently as one would anticipate. For the simple linear advection equation, it is easy to understand. The distance, $A\Delta t$ is the spatial distance that a parcel will travel in time, $\Delta t$, to reach a point, $j\Delta x$. In finite difference form we might approximate our model equation with

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -A(u_j^n - u_{(j-a)}^n)/A\Delta t$$

where $u_{(j-a)}^n = u(n\Delta t, (j-A\Delta t)\Delta x)$. The above equation reduces to

$$u_j^{n+1} = u_j^n(j-a)$$

Unless $A\Delta t$ is an integer number of grid distances, $\Delta x$, we do not know the right-hand side. However, we can use an interpolating polynomial to determine it.

In the two-dimensional problem,

$$u_t + Au_x + Bu_y = 0,$$

we have two distances, $A\Delta t$, and $B\Delta t$ which define a position at some distance from $(j\Delta x, k\Delta y)$. However, the quasi-Lagrangian scheme still reduces to

$$u_{j,k}^{n+1} = u(n\Delta t, j\Delta x-A\Delta t, k\Delta y - B\Delta t)$$

In simple terms, the one term on the right-hand side is the approximation to the entire advection term. Of course, we have to perform a two-dimensional interpolation of $u_{j,k}^n$ to obtain the right-hand side. The scheme is stable if we use the nearest neighbors for the interpolation. The proof is left as an exercise for the reader.

4. STABILITY FOR THE GRAVITY WAVE EQUATIONS

When waves exist in the solution of hyperbolic partial differential equations, the linear stability of the usual second-order finite difference schemes will be controlled by the phase speed of the fastest moving wave. If $C$ is the phase speed, we will find that

$$\frac{C\Delta t}{\Delta x} < 0(1)$$

If advection is included, then the Doppler-shifted speed,
|U| + C governs the stability, i.e.,

\[(|U| + C) \frac{\Delta t}{\Delta x} < 0(1)\]

In this section we investigate the linear stability of a sequence of problems leading to the shallow water equations. Let us consider simple one-dimensional flow for gravity waves. The differential equations are

\[\frac{\partial u}{\partial t} = -g \frac{\partial h}{\partial x}\]

\[\frac{\partial h}{\partial t} = -H \frac{\partial u}{\partial x}\]

Using simple leapfrog and centered differences, we can write

\[u_{j}^{n+1} = u_{j}^{n-1} + \frac{\Delta t}{\Delta x} g(h_{j+1}^{n} - h_{j-1}^{n})\]

\[h_{j}^{n+1} = h_{j}^{n-1} - \frac{\Delta t}{\Delta x} H(u_{j+1}^{n} - u_{j-1}^{n})\]

Define \(\lambda = \frac{\Delta t}{\Delta x}\) and \(C^2 = gH\). Let

\[u_{j}^{n} = U_{n} e^{ikj\Delta x}\]

\[h_{j}^{n} = h_{n} e^{ikj\Delta x}\]

Define \(\alpha = k\Delta x\), and after substituting, we obtain

\[U_{n+1} = U_{n-1} - g\lambda 2i\sin \alpha h_{n}\]

\[h_{n+1} = h_{n-1} - H\lambda 2i\sin \alpha U_{n}\]

If we rewrite the \(U_{n+1}\) equation for \(U_{n+1}\) for \(U_{n+2}\) and \(U_{n}\) and subtract, we can substitute the equation for \(h_{n+1} - h_{n}\). The result is

\[U_{n+2} - 2U_{n} + U_{n+2} = -2i\lambda \cos \alpha (-2i\lambda \sin \alpha U_{n})\]

If an amplification factor, \(G\), exists such that

\[U_{n+2} = GU_{n}\]
Note only every second time
G is

\[ G^2 = (2 - 4\lambda^2 C^2) \]

or

\[ G = 1 - 2\lambda^2 \sin^2 \alpha \pm i\]

If the radical is real, then \(|G| = 1\). The condition can be realized if \(C^2\lambda^2 \sin^2 \alpha \leq 1\), or \(C^2\lambda^2 \leq 1\), which implies that

\[ \frac{C\Delta t}{\Delta x} \leq 1 \]

It should be noted that if \(C\lambda\) exceeds one, then number scales near \(\alpha = \pi/2\) will be the most unstable. These are the small resolved scales,

\[ k = \frac{2\pi}{4\Delta x} \]

The instability will manifest itself in a numerical exponential growth of \(4\Delta x\) waves.

### 4.1. CFL for inertial gravity waves

If we consider the one-dimensional problem

\[ \frac{\partial u}{\partial t} = fv - \frac{\partial h}{\partial x} \]

\[ \frac{\partial v}{\partial t} = -fu \]

\[ \frac{\partial h}{\partial t} = -H \frac{\partial u}{\partial x} \]

and use leapfrog and centered differences, we get

\[ u_j^n = U_n \ e^{ikj\Delta x}, \ v_j^n = V_n \ e^{ikj\Delta x}, \ h_j^n = H_j \]

\[ \alpha = k\Delta x \]

\[ U_{n+1} - U_{n-1} = 2\Delta t f \ V_n - 2i\Delta t \sin \alpha \]

\[ V_{n+1} - V_{n-1} = -2\Delta t \ U_n \]
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\[ h_{n+1} - h_{n-1} = -2iH\sin \alpha \ U_n \]

Elimination of \( h_n \) and \( V_n \) yields

\[ U_{n+2} - 2U_n + U_{n-2} = -4f^2 \Delta t^2 \ U_n - \frac{4gH\lambda^2 \sin^2 \alpha \ U_n}{\Delta r}. \]

Following our standard procedure, we define

\[ U_{n+2} = G U_n \]

which leads to

\[ G^2 - [2-(2f\Delta t)^2 - 4C^2\lambda^2\sin^2\alpha] G + 1 = 0 \]

or

\[ G = \frac{1 - [1 - (2f\Delta t)^2 - 2C^2\lambda^2\sin^2\alpha]^{1/2}}{2} \]

Again, if the radical is real, then \( |G| = 1 \) for all \( \alpha \). This is true if

\[ (f\Delta t)^2 + C^2\lambda^2\sin^2\alpha \leq 1 \]

or

\[ f\Delta t < 1 \] and \( \frac{C\Delta t}{\Delta t} \leq [1 - (f\Delta t)^2]^{1/2} \]

We observe that the Coriolis term, \( f\Delta t \), reduces the CFL condition below unity. In practice, the time step is a small fraction of the inertial period \( 2\pi/f \), and this is not a serious problem.

4.2. C.F.L. for Two-Dimensional Flow

The linearized shallow water equations on an f-plane are

\[ \frac{\partial u}{\partial t} = f v - g \frac{\partial h}{\partial x} \]

\[ \frac{\partial v}{\partial t} = -f v - g \frac{\partial h}{\partial y} \]

\[ \frac{\partial h}{\partial t} = -H\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \]

If we use leapfrog and centered spatial finite differences and we define
the system reduces to

\[ U_{n+1} - U_{n-1} = 2\Delta t V_n - 2ig\lambda_x \sin \alpha h_n \]
\[ V_{n+1} - V_{n-1} = -2\Delta t U_n - 2ig\lambda_y \sin \beta h_n \]
\[ h_{n+1} - h_{n-1} = -2i\lambda_x H \sin \alpha U_n - 2i\lambda_y H \sin \beta V_n \]

where \( \lambda_x = \Delta t/\Delta x, \lambda_y = \Delta t/\Delta y, \alpha = \lambda \Delta x, \) and \( \beta = m \Delta y. \)

In order to eliminate \( V \) and \( h \) in favour of \( U \), we write the first two equations for \( U_{n+1}, U_n \) and \( V_{n+2}, V_n \) to eliminate \( h_n \). These are

\[ U_{n+2} - 2U_n + U_{n-2} = 2\Delta t (V_{n+1} - V_{n-1}) \]
\[-2i\lambda_x g \sin \alpha (-2i\lambda_x H \sin \alpha U_n - 2i\lambda_y H \sin \beta V_n) \]
\[ V_{n+2} - 2V_n + V_{n-2} = -2\Delta t (U_{n+1} - U_{n-1}) \]
\[-2i\lambda_y g \sin \beta (-2i\lambda_x H \sin \alpha U_n - 2i\lambda_y H \sin \beta V_n) \]

Using straightforward linear algebra, we can now eliminate \( V_n \) from these two equations. If we define an amplification factor

\[ U_{n+2} = G U_n \]

we can derive a fourth-order polynomial in \( G \), which fortunately has two roots of magnitude unity. The reduced polynomial is

\[ G^2 + [-2 + 4(f\Delta t)^2 + 4\lambda_x^2 \sin^2 \alpha + 4\lambda_y^2 \sin^2 \beta]G + 1 = 0 \]

or

\[ G^2 - 2AG + 1 = 0 \]

If \((1-A^2)=0\), then \( |G| = 1 \) for all \( \alpha \) and \( \beta \); in other words, the solution is stable for all wavenumbers \( \lambda \) and \( m \). Here

\[ A = 1 - 2(f\Delta t)^2 - 2\lambda_x^2 C^2 \sin^2 \alpha - 2\lambda_y^2 C^2 \sin^2 \beta \]

The stability condition is satisfied if
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\[(f\Delta t)^2 + C_x^2 \sin^2 \alpha + C_y^2 \sin^2 \beta \leq 1\]

for the shortest waves, \(\alpha = \beta = \pi/4\),

\[(f\Delta t)^2 + C_x^2 + C_y^2 \leq 1\]

For the longest waves, \(\alpha = \beta = 0\), the \(|f\Delta t| < 1\).

If \(\Delta x = \Delta y\)

\[\frac{C^2 \Delta t^2}{\Delta x^2} \leq 1 - (f\Delta t)^2\]

Again we note that the Coriolis term, \(f\Delta t\), reduces the CFL condition below unity. It is more important to observe that if \(f = 0\),

\[\frac{C\Delta t}{\Delta x} \leq \frac{\sqrt{2}}{2}\]

In a two-dimensional system a wave can have a velocity of \(C\) in both spatial directions and thus a vector speed of \(\sqrt{2}C\). The linear stability must account for this. As in the one-dimensional case, we observe that if CFL is violated, we expect the \(4\Delta x\) and \(4\Delta y\) waves to grow fastest.

5. CFL FOR TWO-DIMENSIONAL GRAVITY WAVES WITH ADVECTION

The linear primitive equations without rotation are

\[u_t + A u_x + B u_y = -g h_x\]
\[v_t + A v_x + B v_y = -g h_y\]
\[h_t + A h_x + B h_y = -D(u_x + v_y)\]

Let us use second order approximations in time and space and surpress, \(j, k, n\), unless it is indexed

**Define:** \(\lambda_x = \Delta t/\Delta x\), \(\lambda_y = \Delta t/\Delta x\), \(\alpha = \Delta x\), \(\beta = \Delta y\)

\[u^{n+1} = u^{n-1} - A\lambda_x (u_{j+1} - u_{j-1}) - \beta\lambda_y (u_{k+1} - u_{k-1})\]
\[\quad - g\lambda_x (h_{j+1} - h_{j-1})\]

\[v^{n+1} = v^{n-1} - A\lambda_x (v_{j+1} - v_{j-1}) - B\lambda_y (u_{k+1} - u_{k-1})\]
Use our standard linear stability technique and eliminate the spatial dependence

\[(u,v,h) = (U,V,H)e^{im\Delta x} e^{ik\Delta y}\]

The new equations are

\[U_{n+1} = U_{n-1} - A_\lambda x 2i\sin\alpha U - B_\lambda y 2i\sin\beta U - g_\lambda x 2i\sin\alpha H\]
\[V_{n+1} = V_{n-1} - A_\lambda x 2i\sin\alpha V - B_\lambda y 2i\sin\beta V - g_\lambda y 2i\sin\beta H\]
\[H_{n+1} = H_{n-1} - A_\lambda x 2i\sin\alpha H - B_\lambda y 2i\sin\beta H - D_\lambda x 2i\sin\beta V\]

This looks like a difficult algebraic problem. However, the solution is straightforward. Proceed as follows; eliminate \(H\) from the first and third equations and the second and third equations; then eliminate either \(U\) or \(V\) from the two equations; after using

\[U_{n+1} = GU_n\]

we obtain a second-order equation, \(G^2 + 2iRG - 1 = 0\), whose analysis is left to the reader and

\[G^n + 4iRG^2 - 2SG^2 - 4iRG + 1 = 0\]

where

\[R = A\lambda x \sin\alpha + B\lambda y \sin\beta\]
\[S = 1 + 2(A\lambda x \sin\alpha + B\lambda y \sin\beta)^2 - 2gD\lambda x^2 \sin^2\alpha - 2gD\lambda y^2 \sin^2\beta\]

Let \(iX = G\) and obtain

\[X^n + 4RX^2 + 2SX^2 + 4RX + 1 = 0\]

The product of the roots must be unity, and if one, \(|X|<1\) is less than unity, another must be greater in absolute value.

For stability all roots must have magnitude unity. The reader can complete the details. The general condition for stability becomes
THE HYPERBOLIC PROBLEM

\[ |A| \lambda_x + |B| \lambda_y + [gD(\lambda_x^2 + \lambda_y^2)]^{1/2} \leq 1 \]

We can reduce this to the famous CFL condition by letting \( \Delta x = \Delta y \), \( \alpha = \beta = \pi/4 \), \( A = B \) and \( U = (A^2 + B^2)^{1/2} \), to obtain

\[ [U + (gD)^{1/2}] \Delta t/\Delta x \leq \frac{1}{\sqrt{2}} \]

In physical terms, the Doppler-shifted gravity wave may not propagate a greater distance then \( \sqrt{2} \Delta x \) in time \( \Delta t \).

Exercise:

Add the Coriolis terms to the primitive equations with advection and determine the stability condition.

6. COMBINED ADVECTIVE-DIFFUSIVE PROBLEMS

In Chapter 4 we demonstrated that the diffusion equation is unstable if we use a leapfrog scheme. In this chapter, we learned that a forward time step is unstable for the advection equation. In most oceanographical problems we must include some diffusion in the problem; hence it is necessary to combine several time-differencing schemes. The simplest is leapfrog for advection and forward for diffusion. Suppose we have the equation

\[ \frac{\partial q}{\partial t} + A \frac{\partial q}{\partial x} = \frac{\partial^2 q}{\partial x^2} \]

We might use

\[ q_{j+1}^{n+1} = q_j^n - \frac{A \Delta t}{\Delta x} [q_{j+1}^n - q_{j-1}^n] \]
\[ + \frac{2K \Delta t}{(\Delta x)^2} [q_{j+1}^{n-1} + q_{j-1}^{n-1} - 2q_j^{n-1}] \]

Note that the diffusion term is evaluated at \((n-1)\Delta t\). Each term is stable by itself; if \( K = 0 \), then the scheme is stable if \( |A| \Delta t/\Delta x \leq 1 \); if \( A = 0 \), then the scheme is stable if \( K \Delta t/\Delta x^2 \leq 1/4 \). The factor, \( 1/4 \), arises because of the \( 2\Delta t \) time step. In the general case, we get a modified condition. The amplification factor is

\[ G = G^{-1} - 2i(A \Delta t/\Delta x) \sin \alpha + 4(K \Delta t/\Delta x^2)(\cos \alpha - 1)G^{-1} \]

or
\[ G^2 + 21(\Delta t/\Delta x)\sin \alpha G - (1 + 4(\Delta t/\Delta x^2))(\cos \alpha - 1) = 0 \]

The resulting necessary and sufficient condition of numerical stability is

\[ \frac{A^2 \Delta t^2 + 4K \Delta t}{\Delta x^2} \leq 1 \]

One notes that for either A=0 or K=0, the stability conditions of the individual advective and diffusive schemes are recovered, but also that imposing each condition separately is insufficient.

Figure 1. The diagram shows the stability region for the advection-diffusive problem.

We obtain the surprising conclusion that adding explicit diffusion actually reduces the maximum time step allowed for advection. This is not a serious problem in oceanography, since we normally have

\[ \frac{K \Delta t}{(\Delta x)^2} \ll \left( \frac{C \Delta t}{\Delta x} \right)^2, \quad C > 0 \]

for most practical problems.
Exercise:

Derive the linear stability condition for 2-D advection and diffusion using forward-in-time for diffusion and centered-in-time for advection.

Many ocean modellers use DuFort-Frankel for the diffusion term; (10) would be approximated by

\[ q_j^{n+1} = q_j^n - \frac{A\Delta t}{\Delta x} \left[ q_{j+1}^n - q_{j-1}^n \right] \]

\[ + \frac{2K\Delta t}{(\Delta x)^2} \left[ q_{j+1}^n + q_{j-1}^n - q_j^n + q_j^{n+1} - q_j^{n-1} \right] \]

We have combined a conditionally stable advection scheme \(|A|\frac{\Delta t}{\Delta x} < 1\) with an unconditionally stable scheme for diffusion. In this one dimensional case, it can be shown that only the CFL condition need be satisfied. However, in the two-dimensional case, there is a more stringent restriction (Cushman-Roisin, 1984).

Several authors suggest the use of the unstable forward-in-time, centered-in-space (FTCS) advection scheme when diffusion is present. Clancy [1981] derived the necessary and sufficient conditions for stability and suggested the use of this scheme in ocean modelling; (10) is written

\[ q_j^{n+1} = q_j^n - A\Delta t/(2\Delta x)(q_{j+1}^n - q_{j-1}^n) \]

\[ + K\Delta t/\Delta x^2(q_{j+1}^n + q_{j-1}^n - 2q_j^n) \]

Define: \( k = K\Delta t/\Delta x^2 \); \( a = |A|\Delta t/\Delta x \). The amplification factor is

\( G = 1 - \text{iasina} 2k(1 - \cos a) \)

Clancy derives the two conditions for stability. \(|G| \leq 1\) iff

\( K\Delta t/\Delta x \leq 1/2 \)

and

\[ \frac{|A|\Delta x}{2k} \leq 1 \]

The FTCS scheme is not recommended for ocean models, in spite of the enthusiasm of several authors.
7 NONLINEAR STABILITY

A large body of empirical evidence exists which demonstrates that inviscid nonlinear hyperbolic equations will become unstable after many time steps even when the linear stability is not violated. If one attempts an integration of a hyperbolic model with initial conditions which are well resolved at low wavenumber, one finds that after some time, variance will appear on small scales ($\frac{4\Delta x}{2} - 2\Delta x$). The amplitude of the "noise" will grow slowly at first but eventually, at an unpredictable time, it will grow exponentially. This renders the solution useless. Phillips [1959] is credited with the first analytical example which explains this phenomenon. Richtmyer [1963] provided another example which we will reproduce here. Robert et al., [1970] generalized the previous examples.

We approach the example with these preconceptions. In a linear problem, no Fourier mode can interact with any other mode. However when the equations are nonlinear (or have nonconstant coefficients) we expect the modes to interact and create variance in scales which have no variance initially.

Second, we recognize that we have a bandlimited wavenumber space. A uniform grid in Cartesian coordinates can only have wavelengths, $k \in [0, \frac{2\pi}{2\Delta x}]$. If any nonlinear interaction should produce variance in scales, $k > \frac{2\pi}{2\Delta x}$, the grid cannot resolve this energy, and it will be folded into some low wavenumber. Let us arbitrarily call $k < \frac{2\pi}{4\Delta x}$ low wavenumber and $\frac{2\pi}{4\Delta x} < k < \frac{2\pi}{2\Delta x}$ high wavenumber. We expect a priori that even though all initial energy is low wavenumber, nonlinear interactions will eventually provide variance (or energy) at high wavenumbers.

The example of Richtmyer [1963] inspects the stability of a nonlinear problem which has variance at wavelengths, $0$, $4\Delta x$, and $2\Delta x$. The model is

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}$$

where, as in the von Neumann linear analysis, we will not be concerned with any boundary condition, but only perform a local analysis. Let us approximate (1) with a leapfrog scheme

$$u_{j}^{n+1} = u_{j}^{n-} - \frac{\lambda}{2}[(u_{j+1}^{n})^2 - (u_{j-1}^{n})^2]$$

where $\lambda = \Delta t/\Delta x$. It will be useful to write (12) as

$$u_{j}^{n+} = u_{j}^{n-} - \frac{\lambda}{2}[(u_{j+1}^{n} + u_{j}^{n})(u_{j+1}^{n} - u_{j-1}^{n})]$$
We can show that an exact solution for (13) is

\[ u_n^j = C_n \cos \frac{j\pi}{2} + S_n \sin \frac{j\pi}{2} + U_n \cos \frac{j\pi}{2} + V \]

We can identify \((C_n, S_n)\) as the amplitudes of a wave with length, \(4\Delta x\), and \(U_n\) as the amplitude of a wave with length, \(2\Delta x\) and \(V\) as a low wavenumber component with zero wavenumber. When we substitute (14) into (13) we obtain relationships among the amplitudes

\[
\begin{align*}
C_{n+1} - C_{n-1} &= +\lambda S_n (U_n - V) \\
S_{n+1} - S_{n-1} &= -\lambda C_n (U_n + V) \\
U_{n+1} &= U_{n-1}
\end{align*}
\]

The last equation says \(U_n\) may take on different initial values -- say \(A\) and \(B\) -- for the odd and even time steps. If we eliminate \(S_n\) from the first equation, we obtain

\[
C_{n+2} - 2C_n + C_{n-2} = +4\lambda^2(A + V)(B - V)C_n
\]

Is this equation stable in the von Neumann sense? Define \(C_{n+2} = G C_n\) and derive

\[
G^2 - 2(1 + 2\lambda^2(A + V)(B - V))G + 1 = 0
\]

the roots are

\[
G = 1 + 2\lambda^2(A + V)(B - V) \pm \\
\pm \left[ 1 + (1 + 2\lambda^2(A + V)(B - V)(B - V))^2 \right]^{1/2}
\]

If the radical is imaginary, then \(|G| < 1\); this requires the coefficient

\[-1 \leq \lambda^2(A + V)(B - V) \leq 0\]

Obviously this is only possible if \(|A| < V\) and \(|B| < V\). This is violated when the amplitude of the \(2\Delta x\) wave is large. In this case the \(4\Delta x\) wave will grow exponentially and the scheme is unstable.

In numerical models of the ocean or atmosphere, it is important to damp out the smallest space scales in order to control nonlinear instability. This may be done with an explicit eddy viscosity (Chapter 4) or with a dissipative finite difference method.
8. REFERENCES